Assimilation Algorithms
Lecture 2: 3D-Var

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Outline

From Optimal Interpolation to 3D-Var

- In my last lecture, we derived the linear analysis equation

$$\mathbf{x}_a = \mathbf{x}_b + \mathbf{K}\left(\mathbf{y} - \mathcal{H}(\mathbf{x}_b)\right)$$

where

$$\mathbf{K} = \mathbf{P}^b\mathbf{H}^\mathrm{T}\left[\mathbf{H}\mathbf{P}^b\mathbf{H}^\mathrm{T} + \mathbf{R}\right]^{-1} \equiv \left[(\mathbf{P}^b)^{-1} + \mathbf{H}^\mathrm{T}\mathbf{R}^{-1}\mathbf{H}\right]^{-1}\mathbf{H}^\mathrm{T}\mathbf{R}^{-1}$$

- Optimal Interpolation (OI) applies direct solution methods to invert the matrix $\left[\mathbf{H}\mathbf{P}^b\mathbf{H}^\mathrm{T} + \mathbf{R}\right]$.
- Data selection is applied to reduce the dimension of the matrix.
- Direct methods require access to the matrix elements. In particular, $\mathbf{H}\mathbf{P}^b\mathbf{H}^\mathrm{T}$ must be available in matrix form.
- This limits us to very simple observation operators.

From Optimal Interpolation to 3D-Var

- Iterative methods have significant advantages over the direct methods used in OI.
- They can be applied to much larger problems than direct techniques, so we can avoid data selection.
- They do not require access to the matrix elements.
- Typically, to solve $\mathbf{Ax} = \mathbf{b}$, requires only the ability to calculate matrix-vector products: $\mathbf{Ax}$.
- This allows us to use operators defined by pieces of code rather than explicitly as matrices.
- Examples of such operators include radiative-transfer codes, numerical models, Fourier transforms, etc.

Example: Conjugate Gradients

To solve $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A}$ is real, symmetric and positive-definite:

$$\mathbf{r}_0 := \mathbf{b} - \mathbf{A}\mathbf{x}_0 \qquad \mathbf{p}_0 := \mathbf{r}_0 \qquad k := 0$$

repeat until $\mathbf{r}_{k+1}$ is sufficiently small

$$\begin{aligned}
\alpha_k &:= \frac{\mathbf{r}_k^{\mathrm{T}}\mathbf{r}_k}{\mathbf{p}_k^{\mathrm{T}}\mathbf{A}\mathbf{p}_k} \\
\mathbf{x}_{k+1} &:= \mathbf{x}_k + \alpha_k \mathbf{p}_k \\
\mathbf{r}_{k+1} &:= \mathbf{r}_k - \alpha_k \mathbf{A}\mathbf{p}_k \\
\beta_k &:= \frac{\mathbf{r}_{k+1}^{\mathrm{T}}\mathbf{r}_{k+1}}{\mathbf{r}_k^{\mathrm{T}}\mathbf{r}_k} \\
\mathbf{p}_{k+1} &:= \mathbf{r}_{k+1} + \beta_k \mathbf{p}_k \\
k &:= k+1
\end{aligned}$$

The result is x_{k+1}

From Optimal Interpolation to 3D-Var

- There are two ways to apply iterative methods to the linear analysis equation, depending which expression we adopt for $\mathbf{K}$:
- For $\mathbf{K} = \mathbf{P}^b\mathbf{H}^{\mathrm{T}}\left[\mathbf{H}\mathbf{P}^b\mathbf{H}^{\mathrm{T}} + \mathbf{R}\right]^{-1}$ we have:

$$\mathbf{x}_a = \mathbf{x}_b + \mathbf{P}^b\mathbf{H}^{\mathrm{T}}\mathbf{z} \quad \text{where} \quad \left[\mathbf{H}\mathbf{P}^b\mathbf{H}^{\mathrm{T}} + \mathbf{R}\right]\mathbf{z} = \mathbf{y} - \mathcal{H}(\mathbf{x}_b)$$

- For $\mathbf{K} = \left[(\mathbf{P}^b)^{-1} + \mathbf{H}^{\mathrm{T}}\mathbf{R}^{-1}\mathbf{H}\right]^{-1}\mathbf{H}^{\mathrm{T}}\mathbf{R}^{-1}$, we have:

$$\mathbf{x}_a = \mathbf{x}_b + \delta\mathbf{x} \quad \text{where} \quad \left[(\mathbf{P}^b)^{-1} + \mathbf{H}^{\mathrm{T}}\mathbf{R}^{-1}\mathbf{H}\right]\delta\mathbf{x} = \mathbf{H}^{\mathrm{T}}\mathbf{R}^{-1}\left(\mathbf{y} - \mathcal{H}(\mathbf{x}_b)\right)$$

- The first of these alternatives is called PSAS
- The second (although it may not look like it yet) is 3D-Var

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3D-Var

- As we have seen, (linear) 3D-Var analysis can be seen as an application of iterative solution methods to the linear analysis equation.
- Historically, 3D-Var was not developed this way.
- We will now consider this alternative derivation.
- We will need to apply Bayes' theorem:

$$p(A|B) = \frac{p(B|A)p(A)}{p(B)}$$

where $p(A|B)$ is the probability of A given B, etc.

Maximum Likelihood

- We developed the linear analysis equation by searching for a linear combination of observation and background that minimized the variance of the error.
- An alternative approach is to look for the most probable solution, given the background and observations:

$$\mathbf{x}_a = \arg\max_{\mathbf{x}} \left(p(\mathbf{x}|\mathbf{y} \text{ and } \mathbf{x}_b) \right)$$

- It will be convenient to define a cost function

$$J = -\log \left(p(\mathbf{x}|\mathbf{y} \text{ and } \mathbf{x}_b) \right) + const.$$

- Then, since log is a monotonic function:

$$\mathbf{x}_a = \arg\min_{\mathbf{x}} \left(J(\mathbf{x}) \right)$$

Maximum Likelihood

- Applying Bayes' theorem gives:

$$p(\mathbf{x}|\mathbf{y} \text{ and } \mathbf{x}_b) = \frac{p(\mathbf{y} \text{ and } \mathbf{x}_b|\mathbf{x})p(\mathbf{x})}{p(\mathbf{y} \text{ and } \mathbf{x}_b)}$$

- Now, $p(\mathbf{y} \text{ and } \mathbf{x}_b)$ is independent of $\mathbf{x}$.
- *A Priori* we know nothing about $\mathbf{x}$ – all values of $\mathbf{x}$ are equally likely.
- Hence, we can regard $p(\mathbf{x})/p(\mathbf{y} \text{ and } \mathbf{x}_b)$ as independent of $\mathbf{x}$, and write:

$$p(\mathbf{x}|\mathbf{y} \text{ and } \mathbf{x}_b) \propto p(\mathbf{y} \text{ and } \mathbf{x}_b|\mathbf{x})$$

- Furthermore, if observation errors and backgound errors are uncorrelated, then

$$p(\mathbf{y} \text{ and } \mathbf{x}_b|\mathbf{x}) = p(\mathbf{y}|\mathbf{x})p(\mathbf{x}_b|\mathbf{x})$$

$$\Rightarrow \qquad J(\mathbf{x}) = -\log\left(p(\mathbf{y}|\mathbf{x})\right) - \log\left(p(\mathbf{x}_b|\mathbf{x})\right) + const.$$

Maximum Likelihood

- The maximum likelihood approach is applicable to any probability density functions $p(\mathbf{y}|\mathbf{x})$ and $p(\mathbf{x}_b|\mathbf{x})$.
- However, let us consider the special case of Gaussian p.d.f's:

$$\begin{aligned} p(\mathbf{x}_b|\mathbf{x}) &= \frac{1}{(2\pi)^{N/2}|\mathbf{P}_b|^{1/2}} \exp\left[-\frac{1}{2}(\mathbf{x}_b-\mathbf{x})^{\mathrm{T}}(\mathbf{P}_b)^{-1}(\mathbf{x}_b-\mathbf{x})\right] \\ p(\mathbf{y}|\mathbf{x}) &= \frac{1}{(2\pi)^{M/2}|\mathbf{R}|^{1/2}} \exp\left[-\frac{1}{2}(\mathbf{y}-\mathcal{H}(\mathbf{x}))^{\mathrm{T}}\mathbf{R}^{-1}(\mathbf{y}-\mathcal{H}(\mathbf{x}))\right] \end{aligned}$$

- Now, $J(\mathbf{x}) = -\log\left(p(\mathbf{y}|\mathbf{x})\right) - \log\left(p(\mathbf{x}_b|\mathbf{x})\right) + const.$
- Hence, with an appropriate choice of the constant *const*.:

$$J(\mathbf{x}) = \frac{1}{2}(\mathbf{x}_b-\mathbf{x})^{\mathrm{T}}(\mathbf{P}_b)^{-1}(\mathbf{x}_b-\mathbf{x}) + \frac{1}{2}(\mathbf{y}-\mathcal{H}(\mathbf{x}))^{\mathrm{T}}\mathbf{R}^{-1}(\mathbf{y}-\mathcal{H}(\mathbf{x}))$$

- This is the 3D-Var cost function

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Maximum Likelihood

$$J(\mathbf{x}) = \frac{1}{2}\left(\mathbf{x}_b - \mathbf{x}\right)^{\mathrm{T}} \left(\mathbf{P}_b\right)^{-1} \left(\mathbf{x}_b - \mathbf{x}\right) + \frac{1}{2}\left(\mathbf{y} - \mathcal{H}(\mathbf{x})\right)^{\mathrm{T}} \mathbf{R}^{-1} \left(\mathbf{y} - \mathcal{H}(\mathbf{x})\right)$$

- The maximum likelihood analysis corresponds to the global minimum of the cost function
- At the minimum, the gradient of the cost function ($\nabla J(\mathbf{x})$ or $\partial J / \partial \mathbf{x}$) is zero:

$$\nabla J(\mathbf{x}) = \left(\mathbf{P}_b\right)^{-1} \left(\mathbf{x} - \mathbf{x}_b\right) + \mathbf{H}^{\mathrm{T}} \mathbf{R}^{-1} \left(\mathcal{H}(\mathbf{x}) - \mathbf{y}\right) = \mathbf{0}$$

Maximum Likelihood

$$\nabla J(\mathbf{x}) = (\mathbf{P}_b)^{-1}(\mathbf{x} - \mathbf{x}_b) + \mathbf{H}^{\mathrm{T}}\mathbf{R}^{-1}(\mathcal{H}(\mathbf{x}) - \mathbf{y}) = \mathbf{0}$$

- Now, if $\mathcal{H}$ is linear (or if we neglect second-order terms) then

$$\mathcal{H}(\mathbf{x}) = \mathcal{H}(\mathbf{x}_b) + \mathbf{H}(\mathbf{x} - \mathbf{x}_b)$$

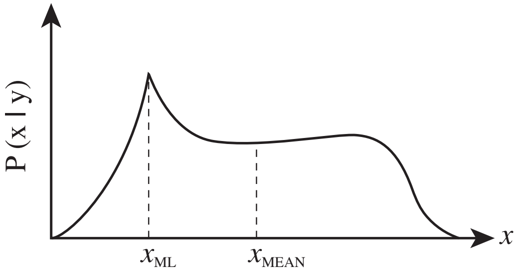
- Hence: $(\mathbf{P}_b)^{-1}(\mathbf{x} - \mathbf{x}_b) + \mathbf{H}^{\mathrm{T}}\mathbf{R}^{-1}(\mathcal{H}(\mathbf{x}_b) + \mathbf{H}(\mathbf{x} - \mathbf{x}_b)) - \mathbf{y}) = \mathbf{0}$
- Rearranging a little gives:

$$\left[(\mathbf{P}_b)^{-1} + \mathbf{H}^{\mathrm{T}}\mathbf{R}^{-1}\mathbf{H}\right]\delta\mathbf{x} = \mathbf{H}^{\mathrm{T}}\mathbf{R}^{-1}(\mathbf{y} - \mathcal{H}(\mathbf{x}_b))$$

  where $\delta\mathbf{x} = \mathbf{x} - \mathbf{x}_b$
- This is exactly the equation for the minimum-variance analysis we derived at the start of the lecture!

Maximum Likelihood

- We have shown that the maximum likelihood approach is naturally expressed in terms of a cost function representing minus the log of the probability of the analysis state.
- The minimum of the cost function corresponds to the maximum likelihood (probability) solution.
- For Gaussian errors and linear observation operators, the maximum likelihood analysis coincides with the minimum variance solution.
- This is not the case in general:



Maximum Likelihood

- In the nonlinear case, the minimum variance approach is difficult to apply.
- The maximum-likelihood approach is much more generally applicable
- As long as we know the p.d.f's, we can define the cost function
  - However, finding the global minimum may not be easy for highly non-Gaussian p.d.f's.
- In practice, background errors are usually assumed to be Gaussian (or a nonlinear transformation is applied to *make* them Gaussian).
  - This makes the background-error term of the cost function quadratic.
- However, non-Gaussian observation errors are taken into account. For example:
  - Directionally-ambiguous wind observations from scatterometers
  - Observations contaminated by occasional gross errors, which make outliers much more likely than implied by a Gaussian model.

Minimization

- In 3D-Var, the analysis is found by minimizing the cost function:

$$J(\mathbf{x}) = \frac{1}{2}\left(\mathbf{x}_b - \mathbf{x}\right)^{\mathrm{T}}\left(\mathbf{P}_b\right)^{-1}\left(\mathbf{x}_b - \mathbf{x}\right) + \frac{1}{2}\left(\mathbf{y} - \mathcal{H}(\mathbf{x})\right)^{\mathrm{T}}\mathbf{R}^{-1}\left(\mathbf{y} - \mathcal{H}(\mathbf{x})\right)$$

- This is a very large-scale ($\dim(\mathbf{x}) \approx 10^8$) minimization problem.
- The size of the problem restricts on the algorithms we can use.
- Derivative-free algorithms (which require only the ability to calculate $J(\mathbf{x})$ for arbitrary $\mathbf{x}$) are too slow.
- This is because each function evaluation gives very limited information about the shape of the cost function.
  - E.g. a finite difference, $J(\mathbf{x} + \delta\mathbf{v}) - J(\mathbf{x}) \approx \delta\mathbf{v}^{\mathrm{T}}\nabla J(\mathbf{x})$, gives a single component of the gradient.
  - We need $O(10^8)$ components to work out which direction is “downhill”.

Minimization

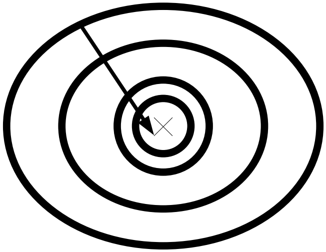
- Practical algorithms for minimizing the 3D-Var cost function require us to calculate its gradient:

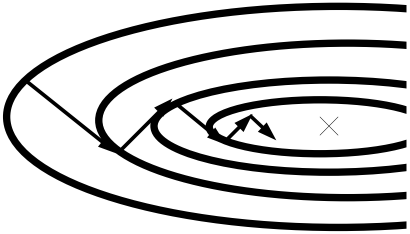
$$\nabla J(\mathbf{x}) = (\mathbf{P}_b)^{-1}(\mathbf{x} - \mathbf{x}_b) + \mathbf{H}^{\mathrm{T}}\mathbf{R}^{-1}(\mathcal{H}(\mathbf{x}) - \mathbf{y})$$

- The simplest gradient-based minimization algorithm is called steepest descent:
  - Let $\mathbf{x}_0$ be an initial guess of the analysis. Repeat the following steps for $k = 0, 1, 2$, etc. until the gradient is sufficiently small:
  - Define a descent direction: $\mathbf{d}_k = -\nabla J(\mathbf{x}_k)$.
  - Find a step α_k, e.g. by line minimization of the function $J(\mathbf{x}_k + \alpha\mathbf{d}_k)$, for which $J(\mathbf{x}_k + \alpha\mathbf{d}_k) < J(\mathbf{x}_k)$.
  - Set $\mathbf{x}_{k+1} = \mathbf{x}_k + \alpha\mathbf{d}_k$.

Minimization

- Steepest descent can work well on very well conditioned problems in which the iso-surfaces of the cost function are nearly spherical.
- In this case, the steepest descent direction points towards the minimum.
- For poorly conditioned problems, with ellipsoidal iso-surfaces, steepest descent is not efficient:





Minimization

- Steepest Descent is inefficient because it does not use information about the curvature (i.e. the second derivatives) of the cost function.
- The simplest algorithm that uses curvature is Newtons method.
- Newton's method uses a local quadratic approximation:

$$J(\mathbf{x} + \delta\mathbf{x}) \approx J(\mathbf{x}) + \delta\mathbf{x}^{\mathrm{T}} \nabla J(\mathbf{x}) + \frac{1}{2} \delta\mathbf{x}^{\mathrm{T}} J'' \delta\mathbf{x}$$

- Taking the gradient gives:

$$\nabla J(\mathbf{x} + \delta\mathbf{x}) \approx \nabla J(\mathbf{x}) + J'' \delta\mathbf{x}$$

- Since the gradient is zero at the minimum, Newton's method chooses the step at each iteration by solving:

$$J'' \delta\mathbf{x} = -\nabla J(\mathbf{x})$$

Minimization

- Newton's method:
  - Start with an initial guess, $\mathbf{x}_0$.
  - Repeat the following steps for $k = 0, 1, 2$, etc. until the gradient is sufficiently small:
  - Solve $J''\delta\mathbf{x}_k = -\nabla J(\mathbf{x}_k)$.
  - Set $\mathbf{x}_{k+1} = \mathbf{x}_k + \delta\mathbf{x}_k$.
- Newton's method works well for cost functions that are well approximated by a quadratic — i.e. for quasi-linear observation operators.
- However, it suffers from several problems. . .
- There is no control on the step length $\|\delta\mathbf{x}\|$. The method can make huge jumps into regions where the local quadratic approximation is poor.
  - This can be controlled using line searches, or by trust region methods that limit the step size to a region where the approximation is valid.

Minimization

- Newton's method requires us to solve $J''\delta\mathbf{x}_k = -\nabla J(\mathbf{x}_k)$ at every iteration.
- Now, J'' is a $\sim 10^8 \times 10^8$ matrix! Clearly, we cannot explicilty construct the matrix, or use direct methods to invert it.
- However, if we have a code that calculates Hessian-vector products, then we can use an iterative method (e.g. conjugate gradients) to solve for $\delta\mathbf{x}_k$.
- Such a code is call a second order adjoint. See Wang, Navon, LeDimet, Zou, 1992 Meteor. and Atmos. Phys. 50, pp3-20 for details.
- Alternatively, we can use a method that constructs an approximation to $(J'')^{-1}$.
- Methods based on approximations of J'' or $(J'')^{-1}$ are called quasi-Newton methods.

Minimization

- By far the most popular quasi-Newton method is the BFGS algorthm, named after its creators Broyden, Fletcher, Goldfarb and Shanno.
- The BFGS method builds up an approximation to the Hessian:

$$\mathbf{B}_{k+1} = \mathbf{B}_k + \frac{\mathbf{y}_k \mathbf{y}_k^{\mathrm{T}}}{\mathbf{y}_k \mathbf{s}_k^{\mathrm{T}}} - \frac{\mathbf{B}_k \mathbf{s}_k \left(\mathbf{B}_k \mathbf{s}_k\right)^{\mathrm{T}}}{\mathbf{s}_k \mathbf{B}_k \mathbf{s}_k^{\mathrm{T}}}$$

  where $\mathbf{s}_k = \mathbf{x}_{k+1} - \mathbf{x}_k$ and $\mathbf{y}_k = \nabla J(\mathbf{x}_{k+1}) - \nabla J(\mathbf{x}_k)$.
- The approximation is symmetric and positive definite, and satisfies

$$\nabla J(\mathbf{x}_{j+1}) - \nabla J(\mathbf{x}_j) = J''(\mathbf{x}_{j+1} - \mathbf{x}_j) \qquad \text{for } j = 0, 1, \cdots, k$$

- There is an explicit expression for the inverse of $\mathbf{B}_k$, which allows Newton's equation to be solved at the cost of $O(Nk)$ operations.

Minimization

- The BFGS quasi-Newton method:
  - Start with an initial guess at the solution, $\mathbf{x}_0$, and an initial approximation of the Hessian (typically,$\mathbf{B}_0 = \mathbf{I}$).
  - Repeat the following steps for $k = 0, 1, 2$, etc. until the gradient is sufficiently small:
  - Solve the approximate Newton's equation, $\mathbf{B}_k \delta\mathbf{x}_k = -\nabla J(\mathbf{x}_k)$, to determine the search direction.
  - Perform a line search to find a step α_k for which for which $J(\mathbf{x}_k + \alpha_k \delta\mathbf{x}_k) < J(\mathbf{x}_k)$.
  - Set $\mathbf{x}_{k+1} = \mathbf{x}_k + \alpha_k \delta\mathbf{x}_k$.
  - Generate an updated approximation to the Hessian: $\mathbf{B}_{k+1}$.
- As k increases, the cost of storing and applying the approximate Hessian increases linearly.
- Moreover, the vectors $\mathbf{s}_k$ and $\mathbf{y}_k$ generated many iterations ago no longer provide accurate information about the Hessian.
- It is usual to construct $\mathbf{B}_k$ from only the $O(10)$ most recent iterations. The algorithm is then called the limited memory BFGS method.

Minimization

- The methods presented so far apply to general nonlinear functions.
- An important special case occurs if the observation operator $\mathcal{H}$ is linear. In this case, the cost function is strictly quadratic, and the gradient is linear:

$$\begin{aligned}\nabla J(\mathbf{x}) &= (\mathbf{P}_b)^{-1}\,\delta\mathbf{x} + \mathbf{H}^{\mathrm{T}}\mathbf{R}^{-1}\left(\mathcal{H}(\mathbf{x}_b) + \mathbf{H}\delta\mathbf{x} - \mathbf{y}\right) \\ &= \left[(\mathbf{P}_b)^{-1} + \mathbf{H}^{\mathrm{T}}\mathbf{R}^{-1}\mathbf{H}\right]\delta\mathbf{x} + \mathbf{H}^{\mathrm{T}}\mathbf{R}^{-1}\left(\mathcal{H}(\mathbf{x}_b) - \mathbf{y}\right)\end{aligned}$$

- In this case, it makes sense to determine the analysis by solving the linear equation $\nabla J(\mathbf{x}) = \mathbf{0}$.
- Since the matrix $\left[(\mathbf{P}_b)^{-1} + \mathbf{H}^{\mathrm{T}}\mathbf{R}^{-1}\mathbf{H}\right]$ is symmetric and positive definite, the best algorithm to use is conjugate gradients. (The algorithm was presented earlier in this lecture.)
- A good introduction to the method can be found online: Shewchuk (1994) "An Introduction to the Conjugate Gradient Method Without the Agonizing pain".

Preconditioning

- We noted that the steepest descent method works best if the iso-surfaces of the cost function are approximately spherical.
- This is generally true of all minimization algorithms.
- In general, expressing the cost function directly in terms of **x** will not lead to spherical iso-surfaces.
- The degree of sphericity of the cost function can be measured by the eigenvalues of the Hessian. (Each eigenvalue corresponds to the curvature in the direction of the corresponding eigenvector.)
- In particular, the convergence rate will depend on the condition number:

$$\kappa = \frac{\lambda_{\max}}{\lambda_{\min}}$$

Preconditioning

- We can speed up the convergence of the minimization by a change of variables $\chi = \mathbf{L}^{-1}(\mathbf{x} - \mathbf{x}_b)$, where $\mathbf{L}$ is chosen to make the cost function more spherical.
- A common choice is $\mathbf{L} = (\mathbf{P}_b)^{1/2}$. The cost function becomes:

$$J(\chi) = \frac{1}{2}\chi^{\mathrm{T}}\chi + \frac{1}{2}\left(\mathbf{y} - \mathcal{H}(\mathbf{x}_b + \mathbf{L}\chi)\right)^{\mathrm{T}} \mathbf{R}^{-1} \left(\mathbf{y} - \mathcal{H}(\mathbf{x}_b + \mathbf{L}\chi)\right)$$

- With this change of variables, the Hessian becomes:

$$J''_{\chi} = \mathbf{I} + \mathbf{L}^{\mathrm{T}}\mathbf{H}^{\mathrm{T}}\mathbf{R}^{-1}\mathbf{H}\mathbf{L} \text{ (plus higher order terms)}$$

- The presence of the identity matrix in this expression guarantees that the minimum eigenvalue is ≥ 1.
- There are no small eigenvalues to destroy the conditioning of the problem.

Calculating the Gradient

- To minimize the cost function, we must be able to calculate gradients.
- If we precondition using **L**, the gradient (with respect to χ) is:

$$\nabla_\chi J(\chi) = \chi + \mathbf{L}^{\mathrm{T}}\mathbf{H}^{\mathrm{T}}\mathbf{R}^{-1}\left(\mathbf{y} - \mathcal{H}(\mathbf{x}_b + \mathbf{L}\chi)\right)$$

- Typically, **R** is diagonal — observation errors are treated as being mutually uncorrelated.
- However, the matrices $\mathbf{H}^{\mathrm{T}}$, $\mathbf{L}^{\mathrm{T}}$ and **L** are not diagonal, and are much too large to be represented explicitly.
- We must represent these as operators (subroutines) that calculate matrix-vector products.

Calculating the Gradient

- Take $\mathcal{H}$ as an example. Each line of the subroutine that applies $\mathcal{H}$ can be considered as a function h_k, so that

$$\mathcal{H}(\mathbf{x}) \equiv h_K\left(h_{K-1}\left(\cdots\left(h_1(\mathbf{x})\right)\right)\right)$$

- Each of the functions h_k can be linearized, to give the corresponding linear function $\mathbf{h}_k$. Each of these is extremely simple, and can be represented by a one or two lines of code.
- The resulting code is called the tangent linear of $\mathcal{H}$.

$$\mathbf{H}(\mathbf{x}) \equiv \mathbf{h}_K \mathbf{h}_{K-1} \cdots \mathbf{h}_1 \mathbf{x}$$

- The transpose, $\mathbf{H}^{\mathrm{T}}(\mathbf{x}) \equiv \mathbf{h}_1^{\mathrm{T}} \mathbf{h}_2^{\mathrm{T}} \cdots \mathbf{h}_K^{\mathrm{T}} \mathbf{x}$, is called the adjoint of $\mathcal{H}$.
- Again, each $\mathbf{h}_k^{\mathrm{T}}$ is extremely simple — just to a few lines of code.
- NB: there is a whole 1-hour lecture on tangent linear and adjoint operators later in the course.

Summary

- We showed that 3D-Var can be considered as an iterative procedure for solving the linear (minimum variance) analysis equation.
- We also derived 3D-Var from the maximum likelihood principle.
- The Maximum Likelihood approach can be applied to non-Gaussian, nonlinear analysis.
- We introduced the 3D-Var cost function.
- We considered how to minimize the cost function using algorithms based on knowledge of its gradient.
- We looked at a simple preconditioning.
- Finally, we saw how it is possible to write code that computes the gradient.