Introduction to chaos

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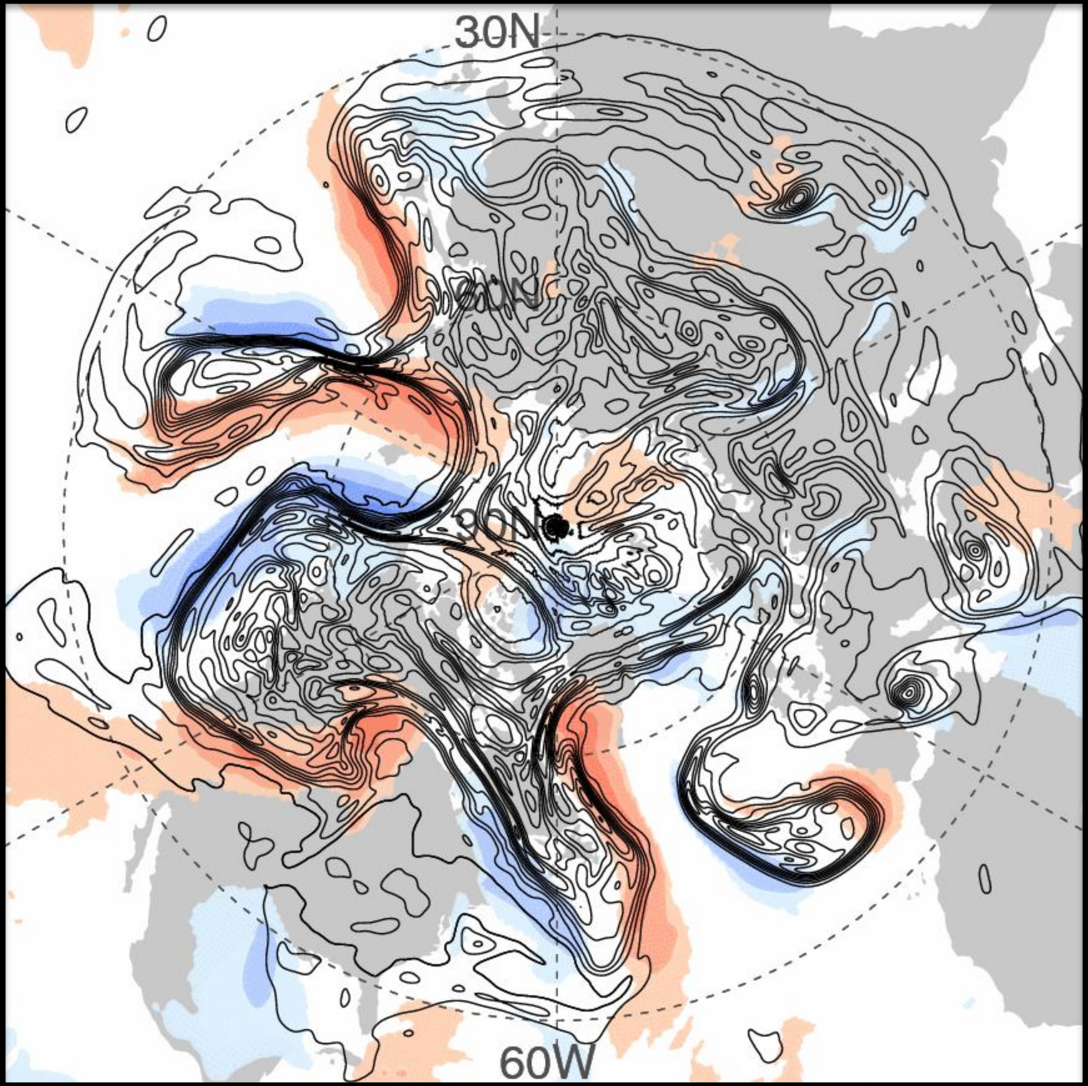

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Vilhelm Bjerknes (1862-1951)

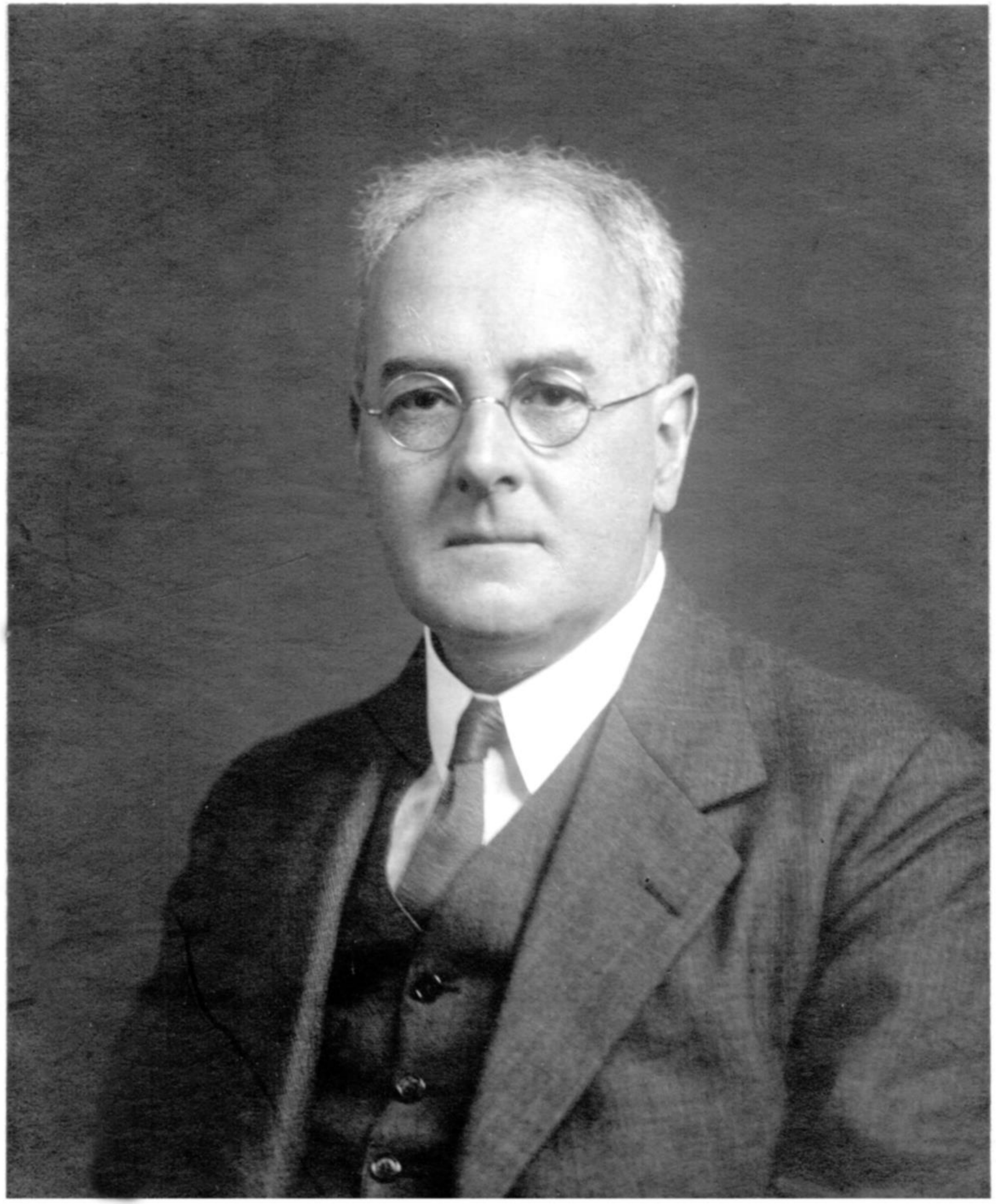
"Founding father of modern weather forecasting"

Norwegian physicist who proposed weather forecasting as a deterministic initial value problem based on the laws of physics

Lewis Fry Richardson (1881-1953)

English scientist who produced the first numerical weather forecast

- Forecast for 20 May 1910 1pm by direct computation of the solutions to simplified flow equations using input data taken at 7am
- Forecast predicted rise in surface pressure by 145 hPa in 6 hours → dramatic failure
- A posteriori: failure to apply smoothing to data to filter out unphysical waves

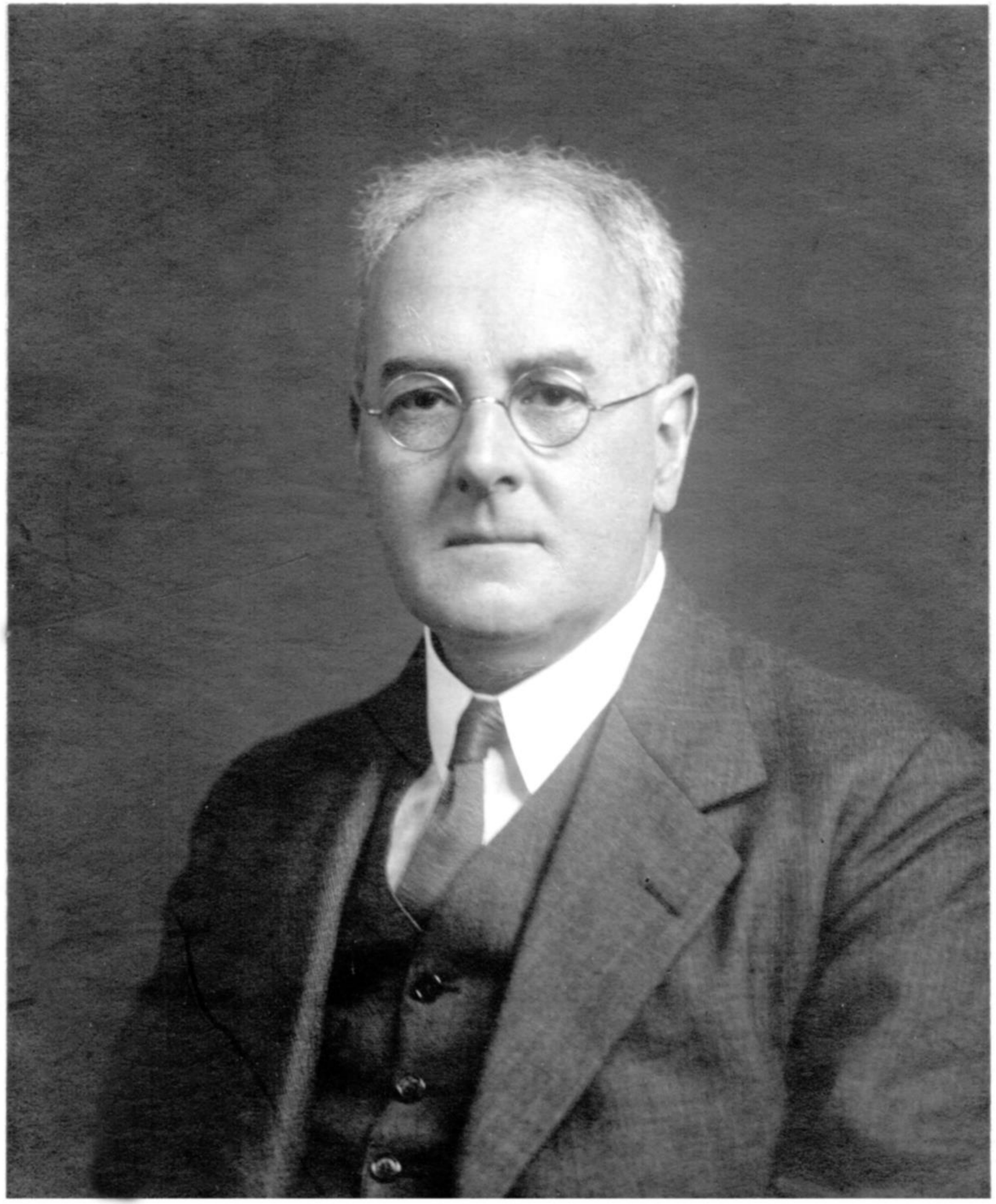


L. F. Richardson, 1931

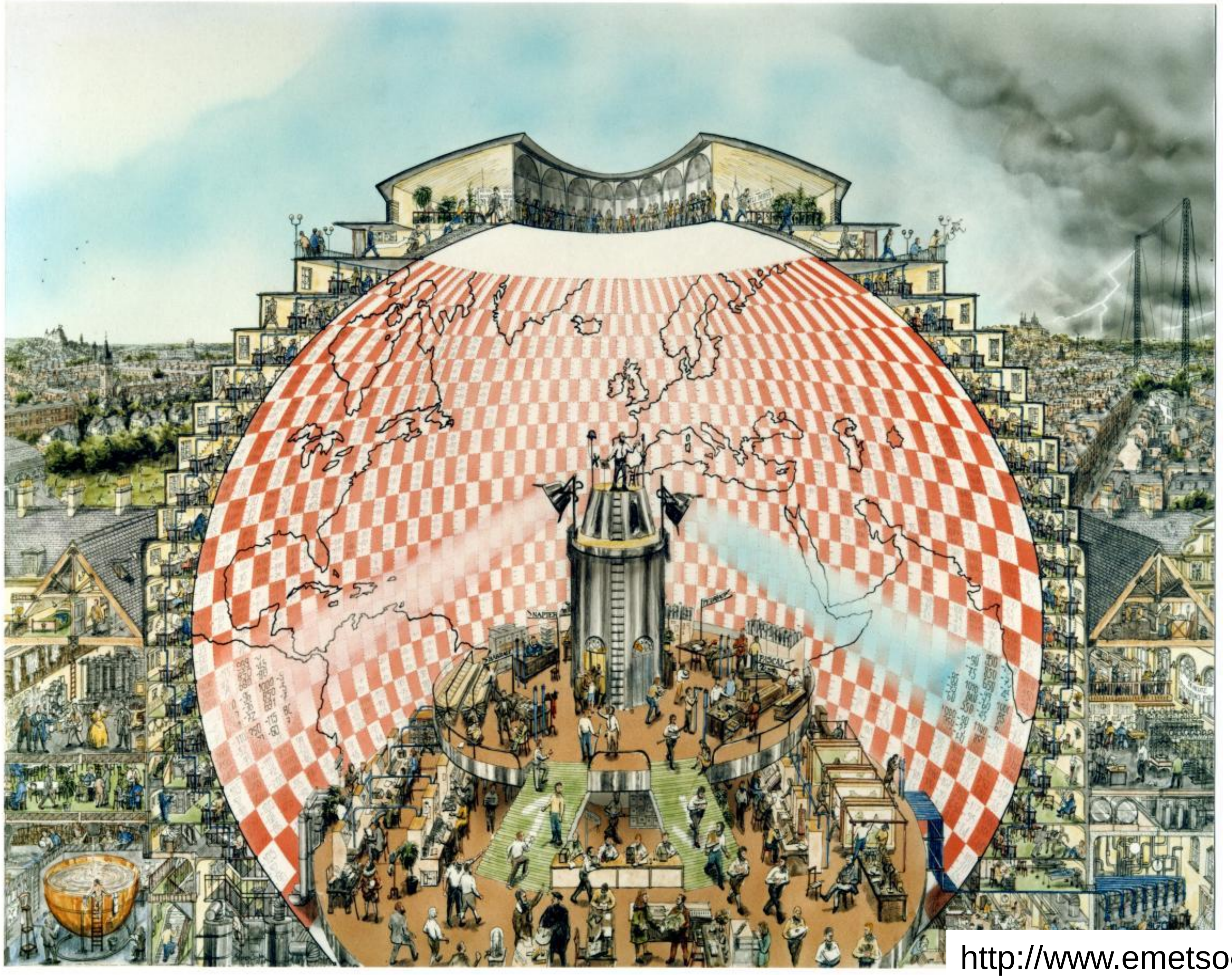
Lewis Fry Richardson (1881-1953)

Author of "Weather Prediction by Numerical Process" (1922)

Richardson devised a method of **solving the mathematical equations** that describe atmospheric flow **by dividing the globe into cells and specifying the dynamical variables at the centre of each cell.** In Chapter 11 of his book, he presents what he calls a 'fantasy', describing in detail **his remarkable vision of an enormous building**, **a fantastic forecast factory**.



L. F. Richardson, 1931



http://www.emetsoc.org/resources/rff

“After so much hard reasoning, may one play with a fantasy? Imagine a large hall like a theatre, except that the circles and galleries go right round through the space usually occupied by the stage. The walls of this chamber are painted to form a map of the globe. The ceiling represents the north polar regions, England is in the gallery, the tropics in the upper circle, Australia on the dress circle and the Antarctic in the pit.

A myriad computers are at work upon the weather of the part of the map where each sits, but each computer attends only to one equation or part of an equation. The work of each region is coordinated by an official of higher rank. Numerous little "night signs" display the instantaneous values so that neighbouring computers can read them. Each number is thus displayed in three adjacent zones so as to maintain communication to the North and South on the map.

From the floor of the pit a tall pillar rises to half the height of the hall. It carries a large pulpit on its top. In this sits the man in charge of the whole theatre; he is surrounded by several assistants and messengers. One of his duties is to maintain a uniform speed of progress in all parts of the globe. In this respect he is like the conductor of an orchestra in which the instruments are slide-rules and calculating machines. But instead of waving a baton he turns a beam of rosy light upon any region that is running ahead of the rest, and a beam of blue light upon those who are behindhand.

Four senior clerks in the central pulpit are collecting the future weather as fast as it is being computed, and dispatching it by pneumatic carrier to a quiet room. There it will be coded and telephoned to the radio transmitting station. Messengers carry piles of used computing forms down to a storehouse in the cellar.

In a neighbouring building there is a research department, where they invent improvements. But there is much experimenting on a small scale before any change is made in the complex routine of the computing theatre. In a basement an enthusiast is observing eddies in the liquid lining of a huge spinning bowl, but so far the arithmetic proves the better way. In another building are all the usual financial, correspondence and administrative offices. Outside are playing fields, houses, mountains and lakes, for it was thought that those who compute the weather should breathe of it freely."

From: L.F. Richardson: Weather Prediction by Numerical Process (1922)

Henry Poincaré (1854-1912)

French mathematician, physicist and philosopher of science

- Fundamental contributions to pure and applied mathematics
- Studying the three-body problem, he became the first person to discover a chaotic deterministic system
- Laid foundations for modern chaos theory

“Why have meteorologists such difficulty in predicting the weather with any certainty? Why is it that showers and even storms seem to come by chance ... a tenth of a degree (C) more or less at any given point, and the cyclone will burst here and not there, and extend its ravages over districts that it would otherwise have spared. **If (the meteorologists) had been aware of this tenth of a degree, they could have known (about the cyclone) beforehand, but the observations were neither sufficiently comprehensive nor sufficiently precise,** and that is the reason why it all seems due to the intervention of chance”

Poincaré, 1909

Sensitive dependence on initial conditions

If we knew exactly the laws of nature and the situation of the universe at the initial moment, we could predict exactly the situation of the same universe at a succeeding moment. But even if it were the case that the natural laws had no longer any secret for us, we could still only know the initial situation *approximately*. If that enabled us to predict the succeeding situation with *the same approximation*, that is all we require, and we should say that the phenomenon had been predicted, that it is governed by laws. But it is not always so; it may happen that **small differences in the initial conditions produce very great ones in the final phenomena**. A small error in the former will produce an enormous error in the latter. Prediction becomes impossible, and we have the fortuitous phenomenon.

Poincaré, 1903 “Science and Method”

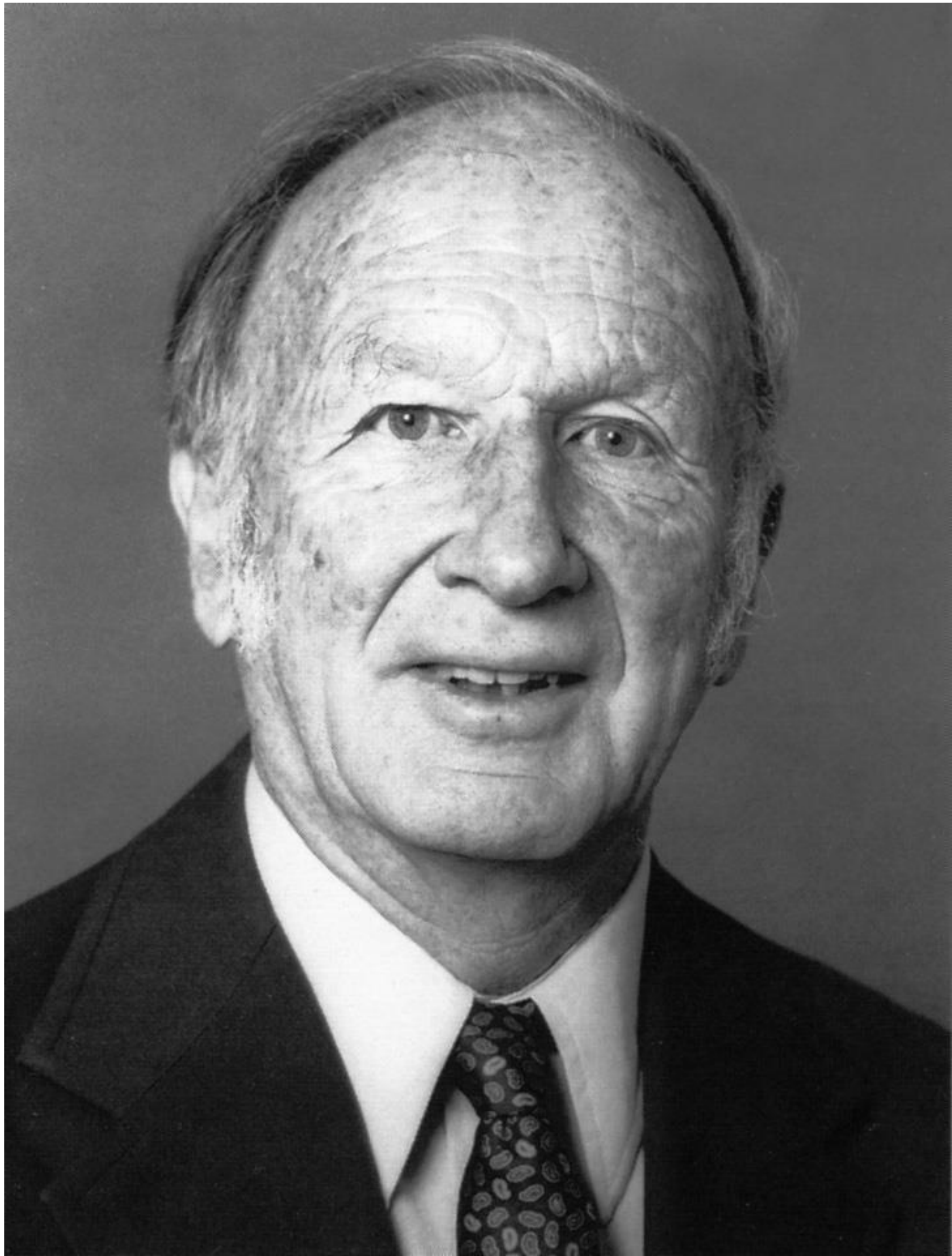
Edward Lorenz (1917 –2008)

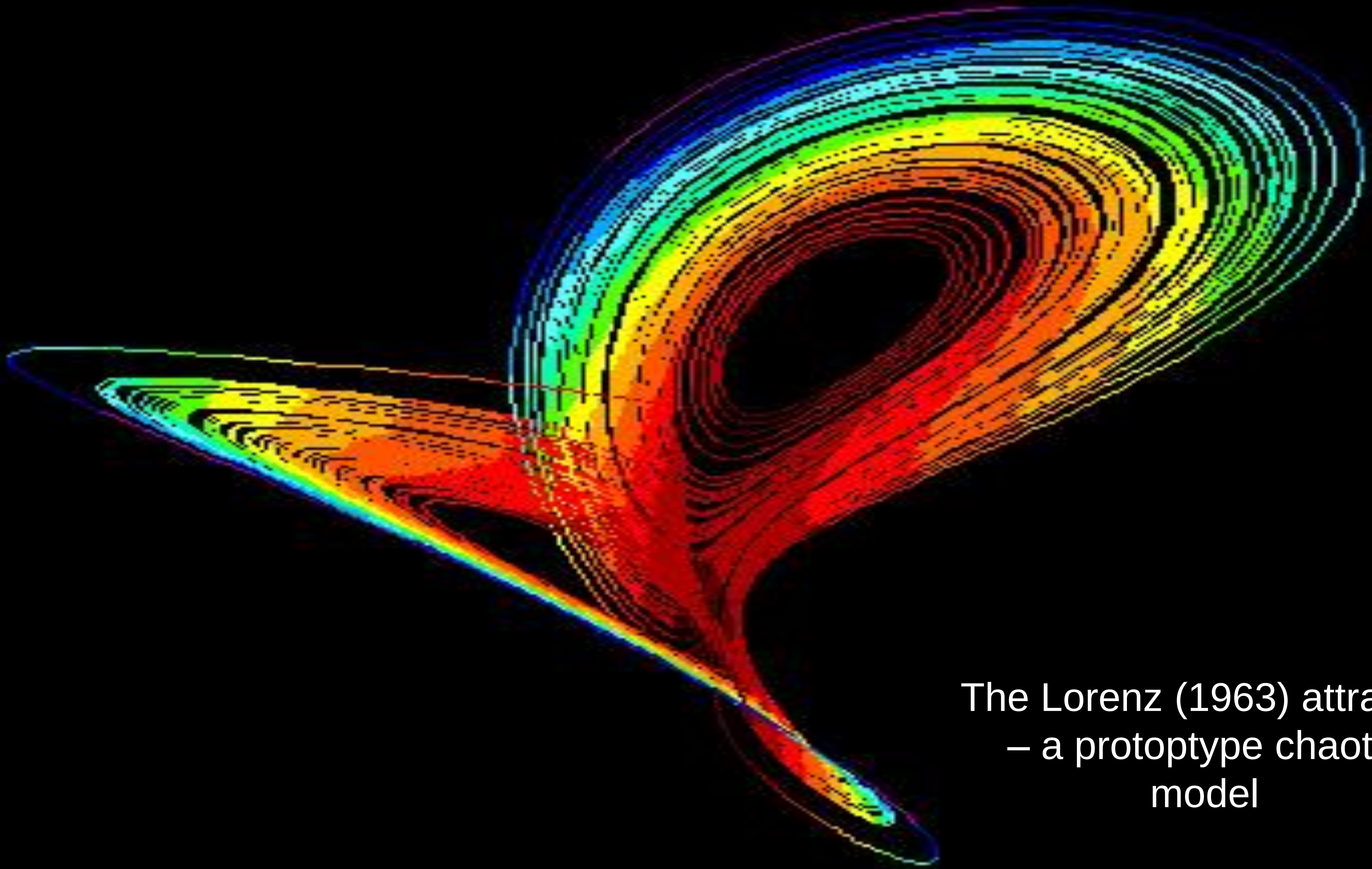
"… one flap of a sea-gull's wing may forever change the future course of the weather" (Lorenz, 1963)

$$\dot{X} = -\sigma X + \sigma Y$$

$$\dot{Y} = -XZ + rX - Y$$

$$\dot{Z} = XY - bZ$$





The Lorenz (1963) attractor – a protoptype chaotic model

What is deterministic chaos?

A physical system that

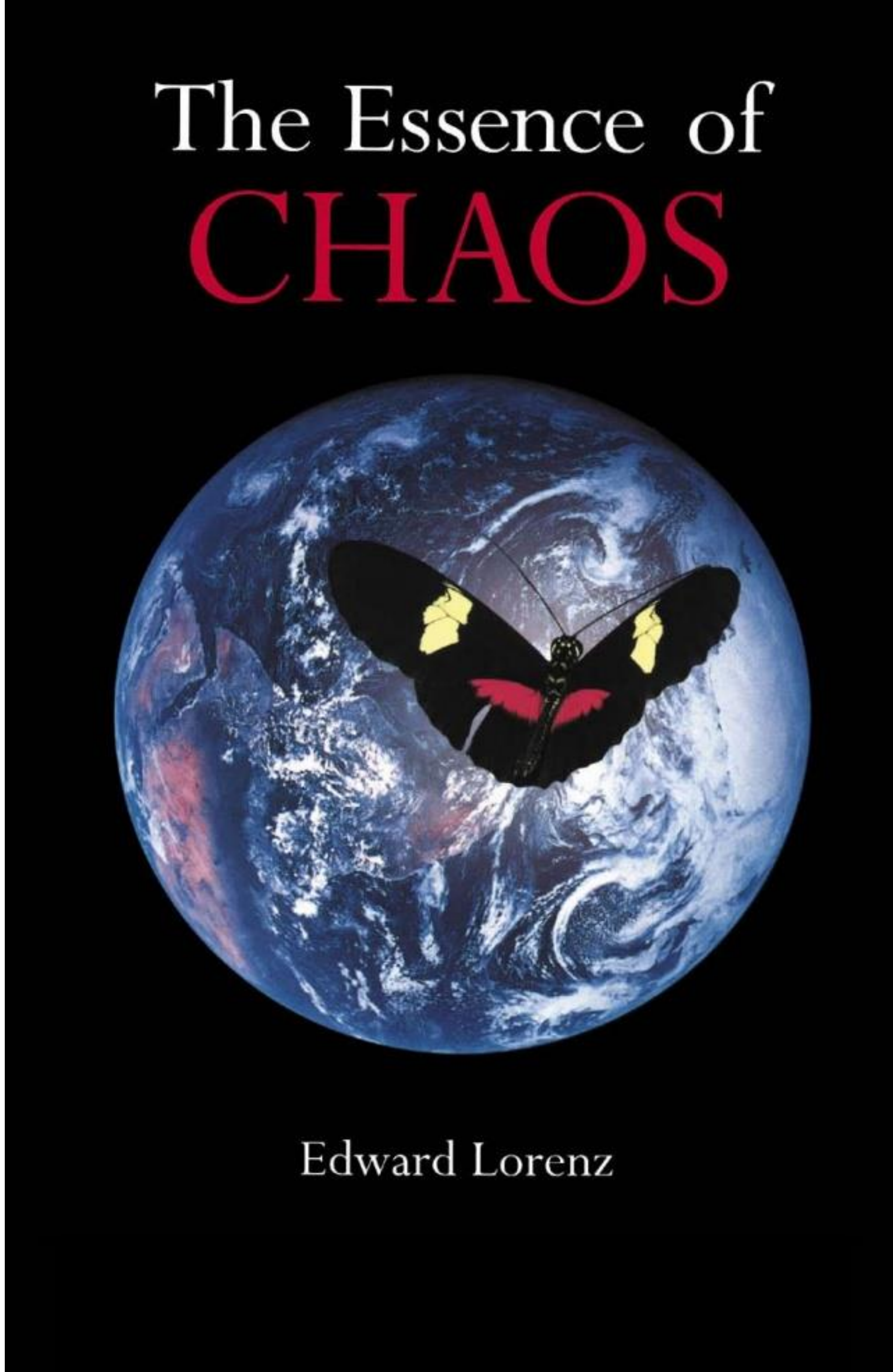
- follows deterministic rules (absence of randomness)
- *but appears* to behave randomly; it *looks* random
- is sensitive dependent on the initial conditions
- needs to be nonlinear, dissipative and at least 3-dimensional
- growth of perturbations is flow dependent

$$\frac{dX}{dt} = F[X] \quad \text{is a nonlinear system}$$

$$\Rightarrow \frac{d\delta X}{dt} = \frac{dF}{dX}\,\delta X \equiv J\ \delta X$$

Since F is a nonlinear function of X

$$\Rightarrow J = J(X)$$



Brief glossary (after E. Lorenz)

Nonlinear system: A system in which alterations in an initial state need not produce proportional alterations in subsequent states

Dissipative system: A dynamical system in which the temporal evolution of any set of points of finite volume in phase space leads to a set of smaller volume

Attractor: In a dissipative system, a limit set that is not contained in any larger limit set, and from which no orbits (trajectories) emanate

Strange attractor: An attractor with a fractal structure (dimension of the set is not a whole number)

Sensitive Dependence: The property characterising an orbit (trajectory) if most other orbits that pass close to it at some point do not remain close to it as time advances

Chaos: The property that characterises a dynamical system in which most orbits (trajectories) exhibit sensitive dependence

Butterfly effect: The phenomenon that a small alteration in the state of a dynamical system will cause subsequent states to differ greatly from the states that would have followed without alteration; sensitive dependence

At this point I want to recount in considerable detail the circumstances that led to my personal involvement with chaos. I readily recall the main events, but my attempts to relate them to earlier and later developments are bound to involve some speculation. The setting was the Department of Meteorology at the Massachusetts Institute of Technology, once familiarly known as Boston Tech, but now almost invariably called M.I.T. I had been doing postdoctoral work there since 1948, and my main interest was the dynamics of atmospheric structures of global and continental size.

As a student I had been taught that the dynamic equations determine what takes place in the atmosphere. However, as my thinking became more and more influenced by numerical weather forecasting, it became evident to me that these equations do not prohibit *any* atmospheric state, realistic or unrealistic, from being an *initial* state in a solution. It must be, I felt, that the various solutions of the equations all converge toward a special set of states — the realistic ones. I had even made a few unsuccessful attempts to find formulas for this set, and had already abandoned the effort. In the light of today's knowledge it appears that I was seeking the attractor, and was right in believing that it existed but wrong in having supposed that it could be described by a few formulas.

The opening scene took place in 1955, when Thomas Malone resigned from our faculty in order to establish and head a new weather research center at the Travelers Insurance Company in Hartford, Connecticut. Tom had been directing a project in statistical weather forecasting, a field that had gained a fair number of adherents in the early days of computers.

Philosophically, statistical forecasting is more like synoptic than dynamic forecasting, in that it is based on observations of what has happened in the past, rather than on physical principles. It is like dynamic forecasting in that it makes use of values of the weather elements at particular locations, rather than identifiable synoptic structures. The type of statistical forecasting that had received most attention was "linear" forecasting, where, for example, tomorrow's temperature at New York might be predicted to be a constant *a,* plus another constant *b* times today's temperature at Chicago, plus another constant *c* times yesterday's relative humidity at St. Louis, plus other similar terms. There were long-established mathematical procedures for estimating the optimum values of the constants *a, b, c,* etc., and, in fact, about the only opportunity for the meteorologist to use any knowledge of the atmosphere was in selecting the predictors—the weather elements to be multiplied by the constants. The computational effort that goes into establishing a formula increases rapidly with the number of predictors, and, as with numerical weather prediction, the work proliferated only after computers became reasonably accessible. The method was regarded by many dynamic meteorologists, particularly those who were championing numerical weather prediction, as a pedestrian approach that yielded no new understanding of why the atmosphere behaved as it did.

I was appointed to fill the vacancy that Tom's departure had created, and with his job I also acquired his project. During the next year I examined numerous statistically derived formulas, and finally convinced myself that what the statistical method was actually doing was attempting to duplicate, by numerical means, what the synoptic forecasters had been doing for many years—displacing each structure at a speed somewhere between its previous speed and its normal speed. One-day prognostic charts were decidedly mediocre, although the method was and still is useful for deciding what local weather conditions to predict, once a prognostic chart is available.

From: E. Lorenz: The Essence of Chaos (1993)

Needless to say, many of the devotees of statistical forecasting disagreed with my findings. Possibly they looked upon me as an infiltrator from the numerical weather prediction camp. In particular, some of them pointed to a recent paper by the eminent mathematician Norbert Wiener, which appeared to show that linear procedures could perform as well as any others, and so necessarily as well as numerical weather prediction or synoptic forecasting. I found this conclusion hard to accept, and convinced myself, although not some of the others, that Wiener's statements, which were certainly correct but were not written in the most easily understandable language, were being misinterpreted. At a meeting in Madison, Wisconsin, in 1956, attended by a large share of the statistical forecasting community, I proposed to test the hypothesis by selecting a system of equations that was decidedly not of the linear type. I would use a computer to generate an extended numerical solution, and then, treating the solution as if it had been a collection of real weather data, I would use standard procedures to determine a set of optimum linear prediction formulas. If these formulas could really match up to any other forecasting scheme, they would have to perform perfectly, since one could easily “predict” the “data” perfectly simply by running the computer program a second time.

My first task was to select a suitable system of equations. I proceeded in the manner of a professional meteorologist and an amateur mathematician. Although in principle a wide variety of systems would have worked, I was hoping to realize some side benefits by choosing a set of equations resembling the ones that describe the behavior of the atmosphere. After some experimentation I decided to work with a drastically simplified form of the filtered equations of numerical weather forecasting, which would reduce the number of variables from the many thousands generally used to a mere handful.

One day Robert White, a postdoctoral scientist in our department who later went on to become Chief of the United States Weather Bureau, and still later headed the organizations that superseded it, suggested that I acquire a small computer to use in my office. If you wonder why I had not already done so, recall that this was more than twenty years before personal computers first appeared on the market. In fact, computers for personal use were almost unheard of, and the idea had certainly not occurred to me. We spent several months considering various competing models and finally settled upon a Royal-McBee LGP-30, which was about the size of a large desk and made a continual noise. It had an internal memory of 4096 32-bit words, of which about a third had to be reserved for standard input and output programs. It performed a multiplication in 17 milliseconds and printed a full line of numbers in about 10 seconds. Even so, when programmed in optimized machine language it was about a thousand times as fast as a desk calculator—pocket calculators had not yet appeared—and was ideal for solving small systems of equations.

It should not surprise us that in a day when computers were far from ubiquitous, most scientists, myself included, had not learned to write computer programs. I spent the next few months getting acquainted with the computer. Upon returning to the simplified meteorological equations, I settled on a form with fourteen variables. Later I cut the number to thirteen and then to twelve by suppressing the variations of one and then two of the variables.

From: E. Lorenz: The Essence of Chaos (1993)

The equations contained several constants that specified the intensity and distribution of the external heating needed to drive the miniature atmosphere. Thus, if one set of constants failed to produce a useful solution, there were always others to try. My early attempts to generate “data” invariably produced “weather” that settled down to a steady state and was therefore useless for my purposes. After many experiments, I at last found a solution that unmistakably simulated the vacillation observed in the dishpan. I eagerly turned to the procedure for determining the best linear formula, only to realize that *perfect* linear prediction was possible simply by predicting that each variable would assume the value that it had assumed one vacillation cycle earlier. It was then that I recognized that for my test I would need a set of equations whose solutions were not periodic. What I did not even suspect at the time was that any such set would have to exhibit sensitive dependence.

By this time it was 1959. Although by now I had become a part of the statistical forecasting community, I managed to retain my status as a dynamicist, and I planned to attend a symposium in numerical weather prediction to be held the following year in Tokyo. Titles for the talks were due well in advance. I gambled on finding a suitable system of equations and completing my test, and submitted the title “The Statistical Prediction of Solutions of Dynamic Equations.”

If I had been familiar then with Poincaré’s work in celestial mechanics, it might have made sense for me to abandon the twelve equations and turn to the four equations of Hill’s reduced problem, which, besides already being known to possess some nonperiodic solutions, were a good deal simpler. My guess, though, is that such a switch would not have appealed to me; the mere knowledge that simple systems with nonperiodic solutions did exist might have given me additional encouragement to continue my own search, and in any case I still had my eye on the possible side benefits. These, I felt, demanded that I work with a dissipative system. As it was, I kept trying new combinations of constants, and finally encountered the long-sought nonperiodic behavior after making the external heating vary with longitude as well as latitude. This is of course what happens in the real atmosphere, which, instead of receiving most of its heat directly from the sun, gets it from the underlying oceans and continents after they have been heated by the sun. Continents and oceans differ considerably in their capacity to absorb solar energy, and in the manner in which they subsequently transfer it to the atmosphere. When I applied the standard procedure to the new “data” the resulting linear forecasts were far from perfect, and I felt that my suspicions had been confirmed.

The solutions proved to be interesting in their own right. The numerical procedure advanced the weather in six-hour increments, and I had programmed the computer to print the time, plus the values of the twelve, thirteen, or fourteen variables, once a day, or every fourth step. Simulating a day required about one minute. To squeeze the numbers onto a single line I rounded them off to three decimal places, and did not print the decimal points. After accumulating many pages of numbers, I wrote an alternative output program that made the computer print one or two symbols on each line, their distances from the margin indicating the values of one or two chosen variables, and I would often draw a continuous curve through successive symbols to produce a graph. It was interesting to watch the graph extend itself, and we would sometimes gather around the computer and place small bets on what would happen next, just as meteorologists often bet on the next day’s real weather. We soon learned

From: E. Lorenz: The Essence of Chaos (1993)

some of the telltale signs for peculiar behavior; in effect, we were learning to be synoptic forecasters for the makebelieve atmosphere.

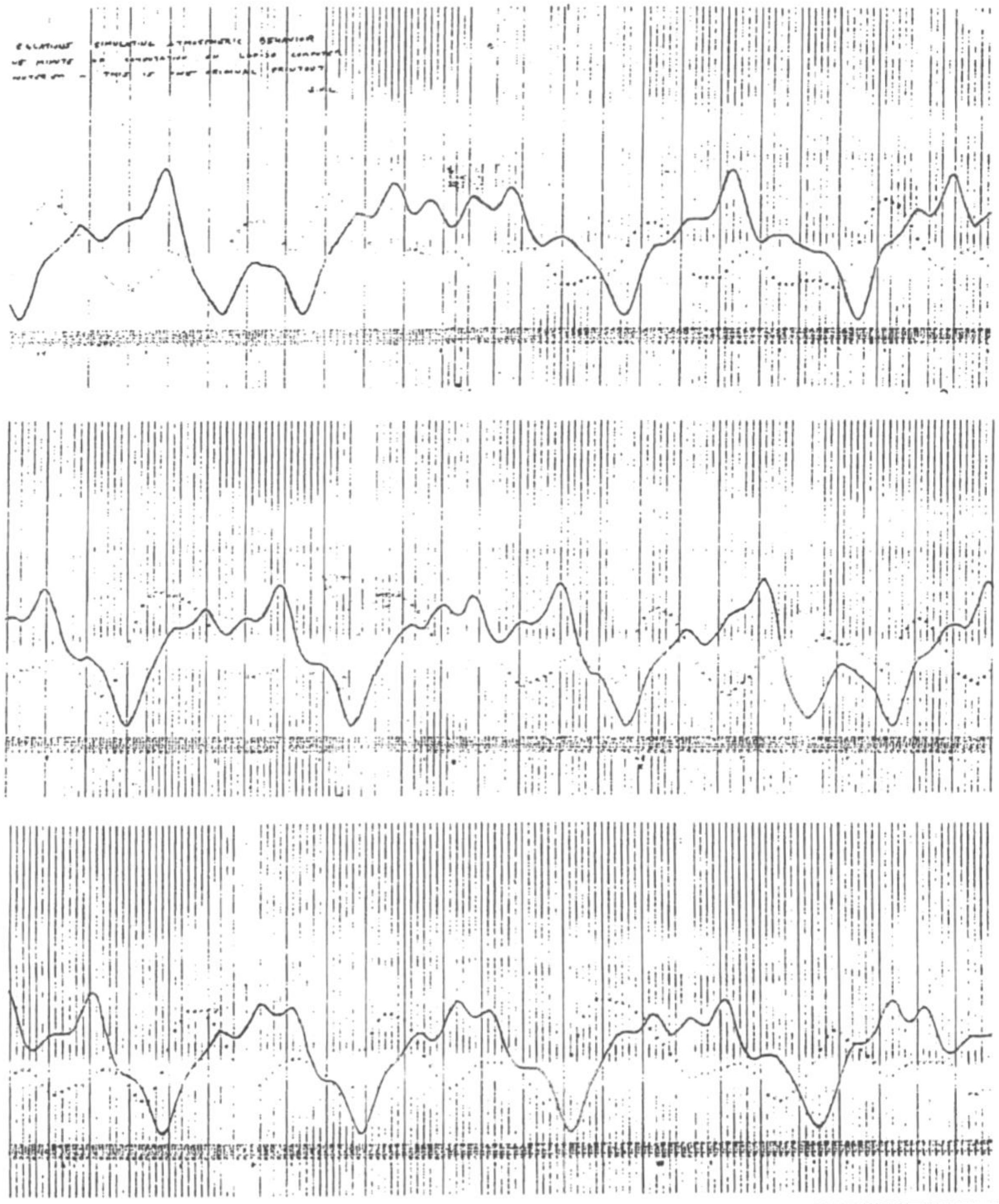
In Figure 43 we see a copy of fifteen months of the somewhat faded original output, divided for display purposes into three fivemonth segments. The chosen variable is an approximate measure of the latitude of the strongest westerly winds; a high value indicates a low latitude. There is a succession of “episodes,” in each of which the value rises abruptly, remains rather high for a month or so, and then drops equally abruptly, but the episodes are not identical and are not even equal in length, and the behavior is patently nonperiodic.

At one point I decided to repeat some of the computations in order to examine what was happening in greater detail. I stopped the computer, typed in a line of numbers that it had printed out a while earlier, and set it running again. I went down the hall for a cup of coffee and returned after about an hour, during which time the computer had simulated about two months of weather. The numbers being printed were nothing like the old ones. I immediately suspected a weak vacuum tube or some other computer trouble, which was not uncommon, but before calling for service I decided to see just where the mistake had occurred, knowing that this could speed up the servicing process. Instead of a sudden break, I found that the new values at first repeated the old ones, but soon afterward differed by one and then several units in the last decimal place, and then began to differ in the next to the last place and then in the place before that. In fact, the differences more or less steadily doubled in size every four days or so, until all resemblance with the original output disappeared somewhere in the second month. This was enough to tell me what had happened: the numbers that I had typed in were not the exact original numbers, but were the rounded-off values that had appeared in the original printout. The initial roundoff errors were the culprits; they were steadily amplifying until they dominated the solution. In today’s terminology, there was chaos.

It soon struck me that, if the real atmosphere behaved like the simple model, long-range forecasting would be impossible. The temperatures, winds, and other quantities that enter our estimate of today’s weather are certainly not measured accurately to three decimal places, and, even if they could be, the interpolations between observing sites would not have similar accuracy. I became rather excited, and lost little time in spreading the word to some of my colleagues.

In due time I convinced myself that the amplification of small differences was the *cause* of the lack of periodicity. Later, when I presented my results at the Tokyo meeting, I added a brief description of the unexpected response of the equations to the round-off errors.

From: E. Lorenz: The Essence of Chaos (1993)



A fifteen-month section of the original print-out of symbols representing two variables of the twelve-variable model. A solid curve had been drawn through the symbols for one variable, while the symbols for the other are faintly visible. The section has been broken into three five-month segments, shown on consecutive rows.

From: E. Lorenz: The Essence of Chaos (1993)

Predictability: Does a flap of a butterfly's wings in Brazil set off a tornado in Texas?

- Talk by Ed Lorenz at a GARP session in Washington, D.C. on 29 December 1972 -

"Lest I appear frivolous in even posing the title question, let alone suggesting that it might have an affirmative answer, let me try to place it in proper perspective by offering two propositions.

1. **If a single flap of a butterfly's wings can be instrumental in generating a tornado, so also can all the previous and subsequent flaps of its wings, as can the flaps of the wings of millions of other butterflies, not to mention the activities of innumerable more powerful creatures, including our own species.**

2. **If the flap of a butterfly's wings can be instrumental in generating a tornado, it can equally well be instrumental in preventing a tornado."**

From: E. Lorenz: The Essence of Chaos (1993)

Predictability: Does a flap of a butterfly's wings in Brazil set off a tornado in Texas?

- Talk by Ed Lorenz at a GARP session in Washington, D.C. on 29 December 1972 -

Although we cannot claim to have proven that the atmosphere is unstable, the evidence that it is so is overwhelming. The most significant results are the following.

1. **Small errors in the coarser structure of the weather pattern**—those features which are readily resolved by conventional observing networks—**tend to double in about three days. As the errors become larger the growth rate subsides.** This limitation alone would allow us to extend the range of acceptable prediction by three days every time we cut the observation error in half, and would offer the hope of eventually making good forecasts several weeks in advance.
2. **Small errors in the finer structure**—e.g., the positions of individual clouds— **tend to grow much more rapidly, doubling in hours or less.** This limitation alone would not seriously reduce our hopes for extended-range forecasting, since ordinarily we do not forecast the finer structure at all.
3. **Errors in the finer structure, having attained appreciable size, tend to induce errors in the coarser structure.** This result, which is less firmly established than the previous ones, implies that after a day or so there will be appreciable errors in the coarser structure, which will thereafter grow just as if they had been present initially. **Cutting the observation error in the finer structure in half**—a formidable task—**would extend the range of acceptable prediction of even the coarser structure only by hours or less**. The hopes for predicting two weeks or more in advance are thus greatly diminished.
4. **Certain special quantities such as weekly average temperatures and weekly total rainfall may be predictable at a range at which entire weather patterns are not**.

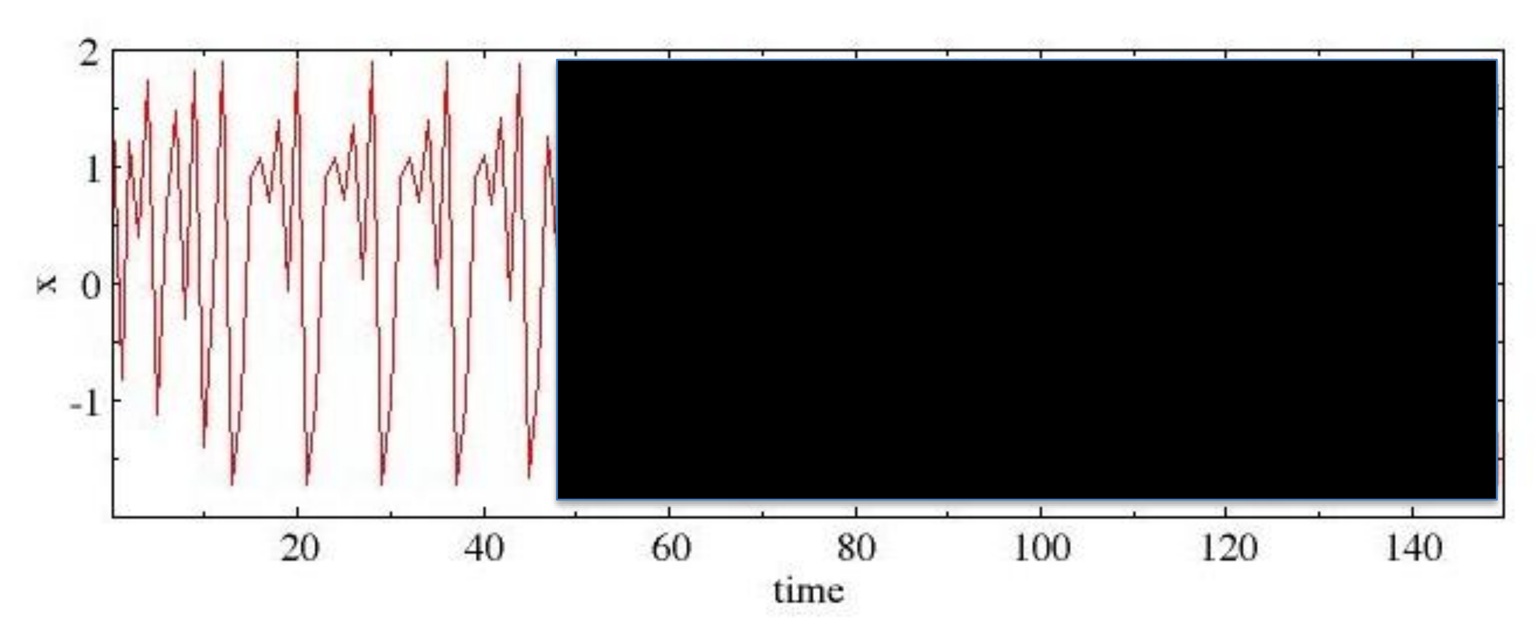
From: E. Lorenz: The Essence of Chaos (1993)

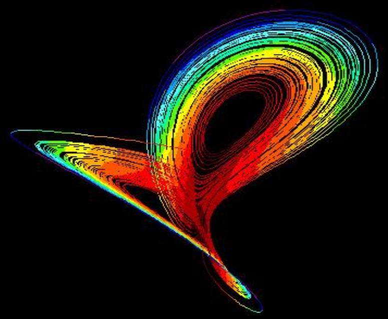
Ed Lorenz (1963): Deterministic Nonperiodic Flow

Dynamical system that is highly sensitive to perturbations of the initial conditions

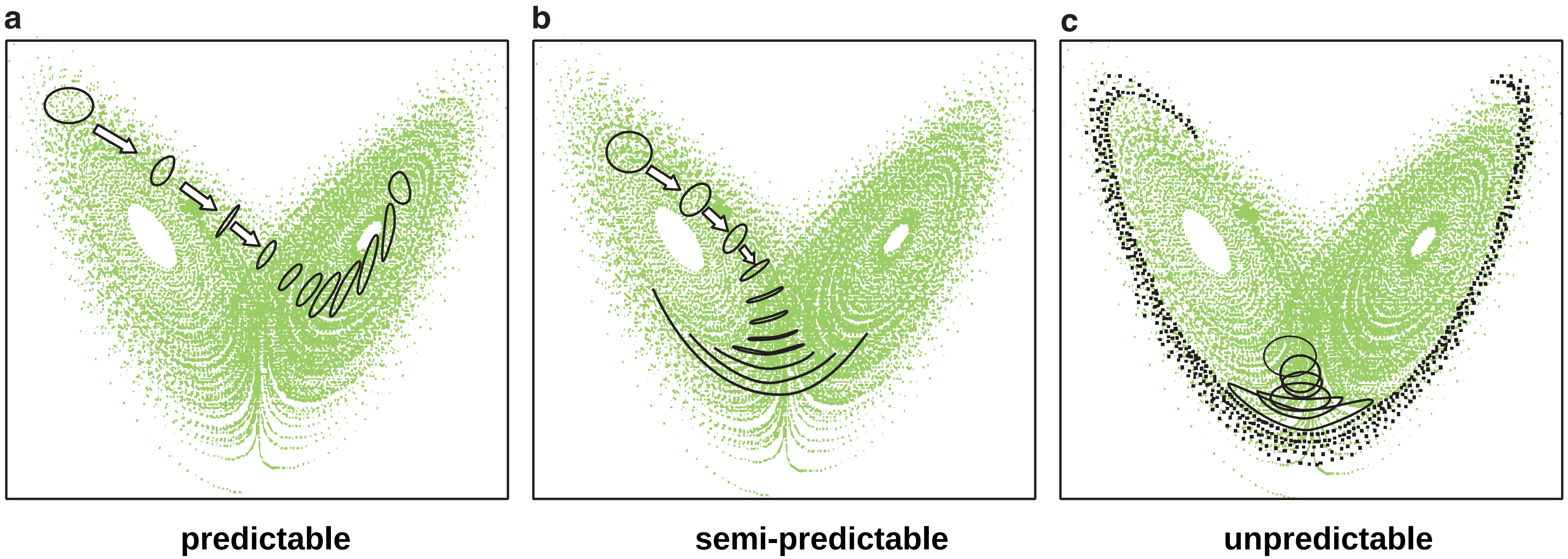
(deterministic chaos)

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} -\sigma(x-y) \\ (r-z)x-y \\ xy-bz \end{pmatrix}$$



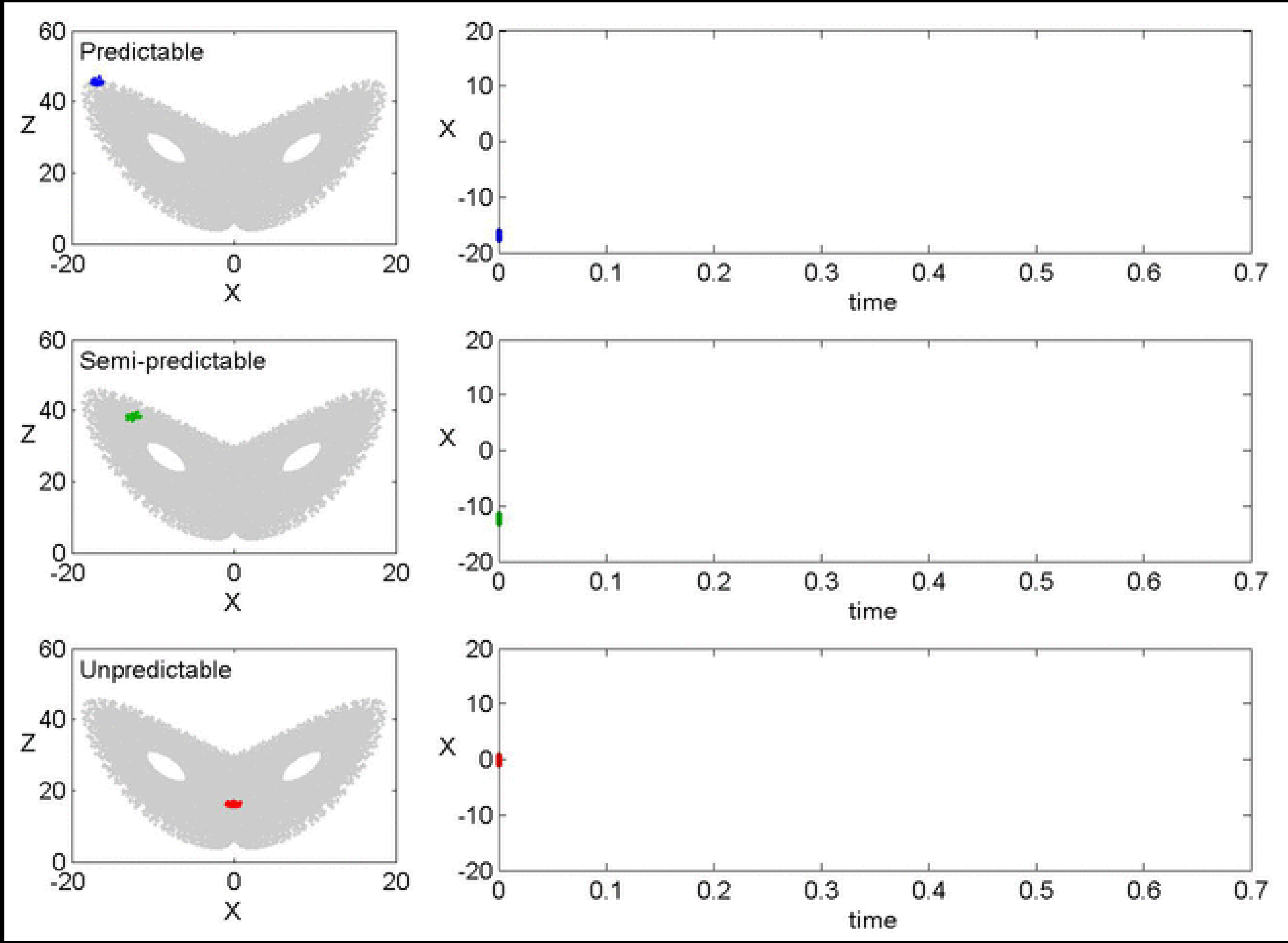


In a nonlinear system the growth of initial uncertainty is flow dependent.



The set of initial conditions (black circle) is located in different regions of the attractor in a), b) and c) and leads to different error growth and predictability in each case.

Lorenz (1963)



Courtesy Hannah Christensen

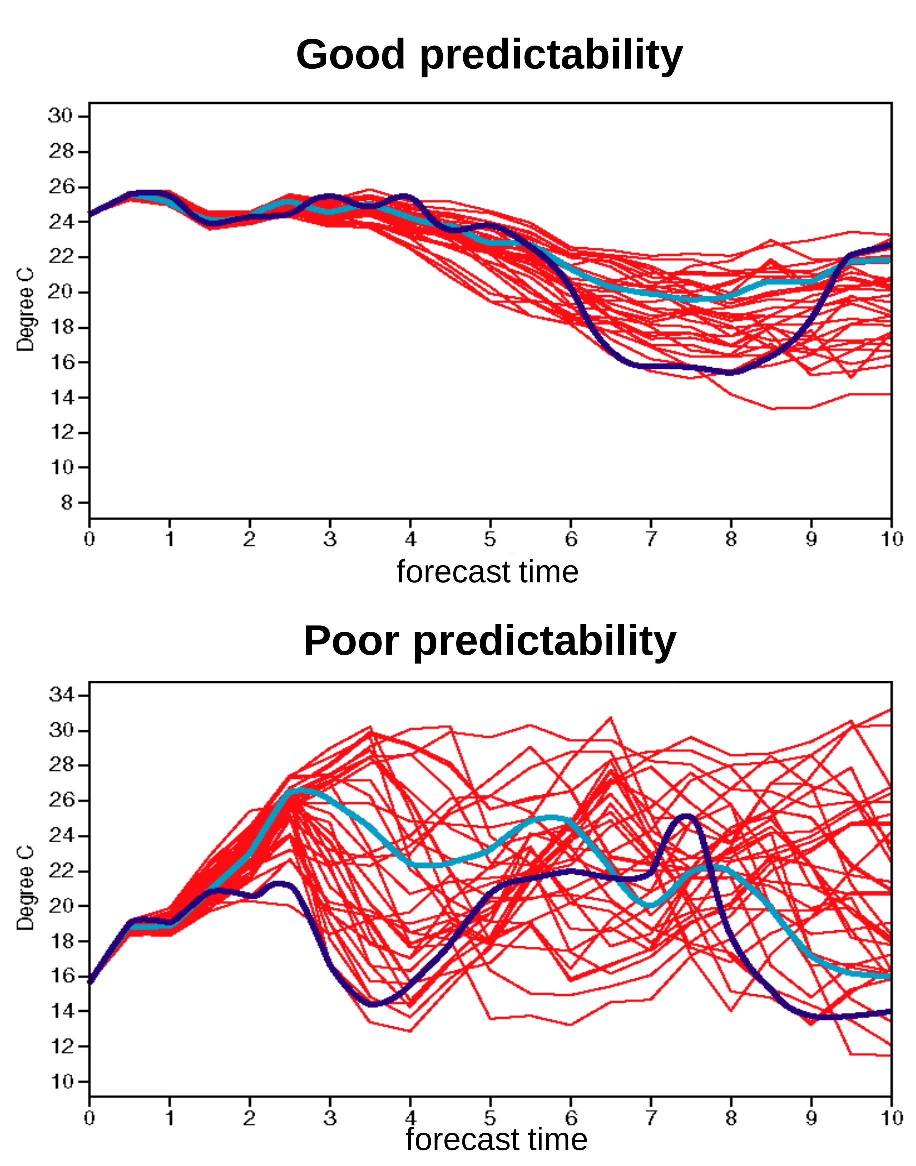
Chaos and ensemble forecasting

The climate is a chaotic system where the future state of the system can be very sensitive to small differences in the current (initial) state of the system.

In practice, the initial state of the system is always uncertain.

Our forecast models are not perfect in all aspects (e.g. small-scale features such as clouds).

Ensemble forecasting takes into account these inherent uncertainties by running a large number of similar but not identical versions of the model in parallel. The resulting forecasts are expressed in probabilities.

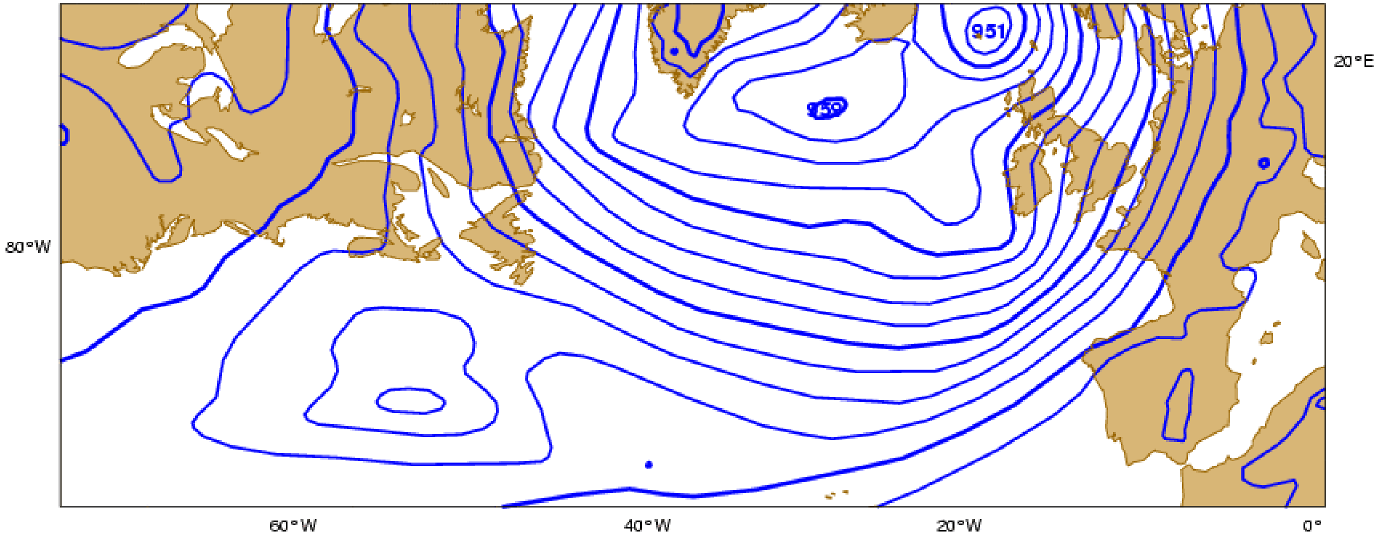


Lothar: 08Z, 26 Dec. 1999

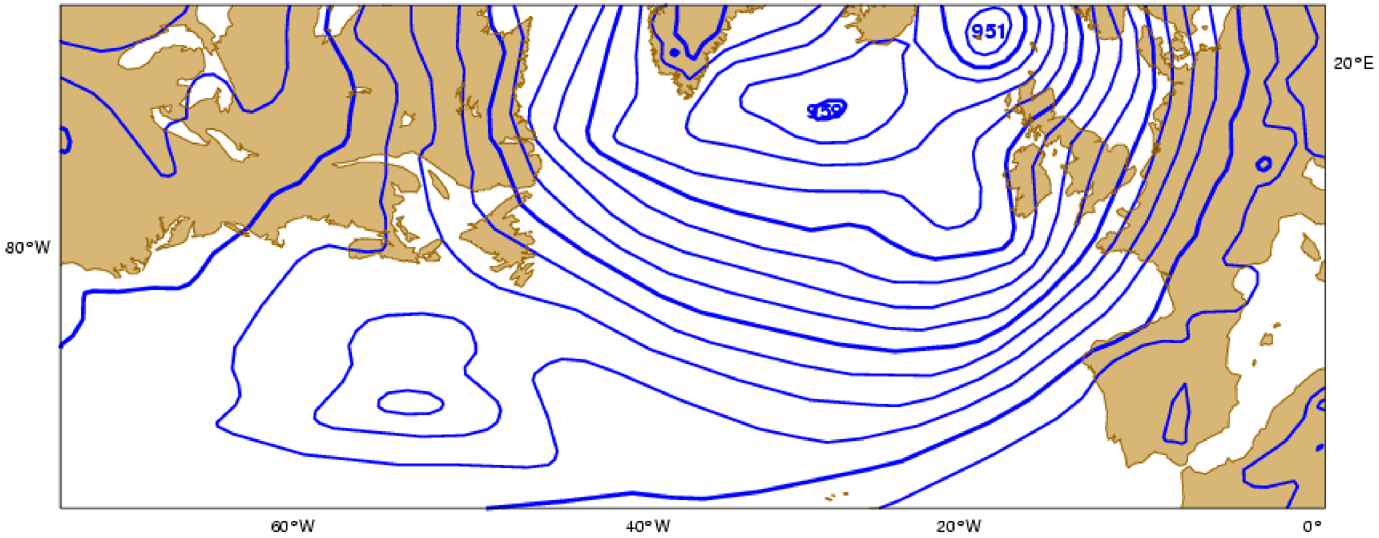
Storms Lothar +Martin

- 100 fatalities
- 400 million trees blown down
- 3.5 million electricity users affected for 20 days
- 3 million people without water

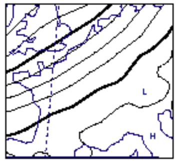
forecast 24 December 1999, 12UTC +0 h - mem no. 13 of 17

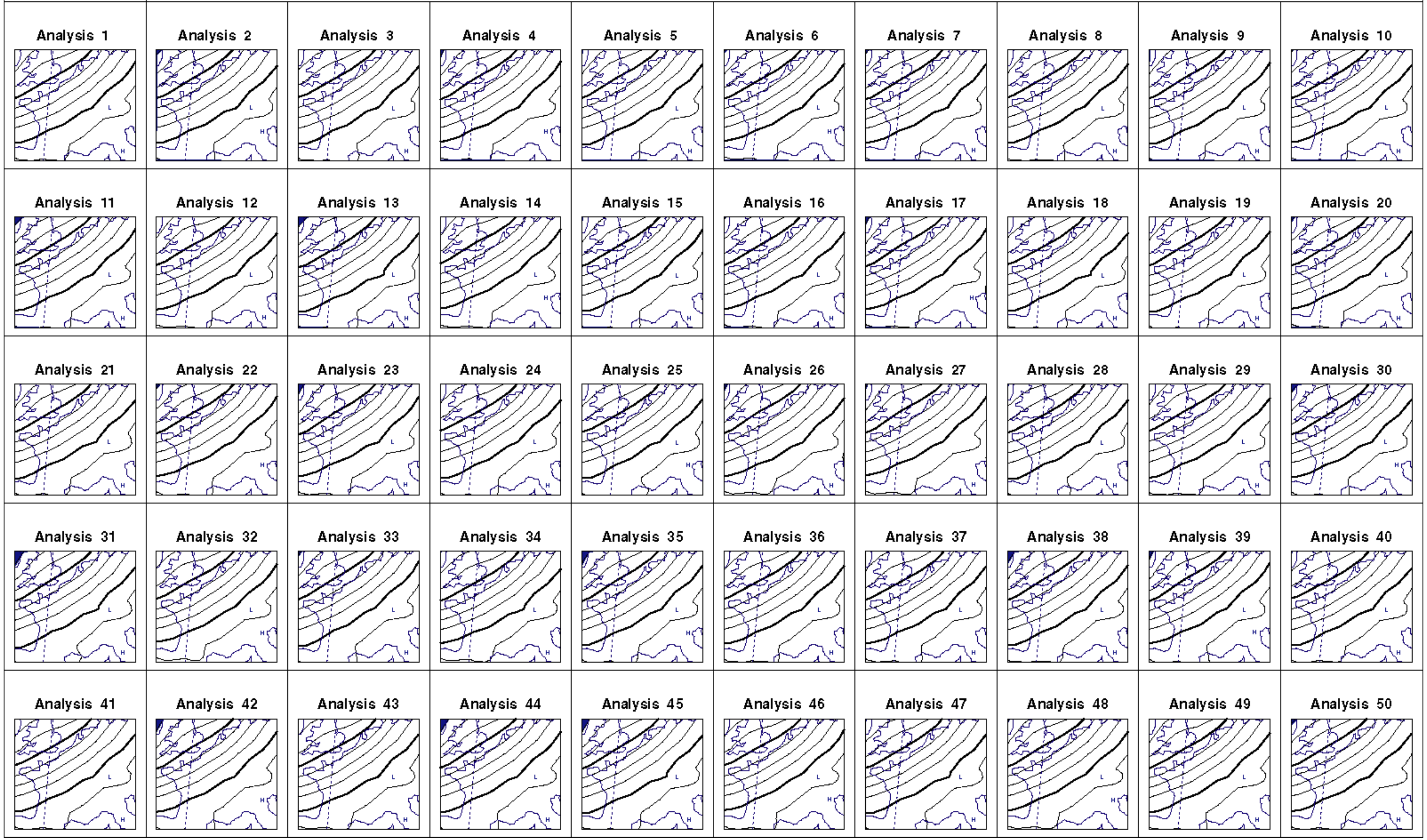


forecast 24 December 1999, 12UTC +0 h - mem no. 14 of 17

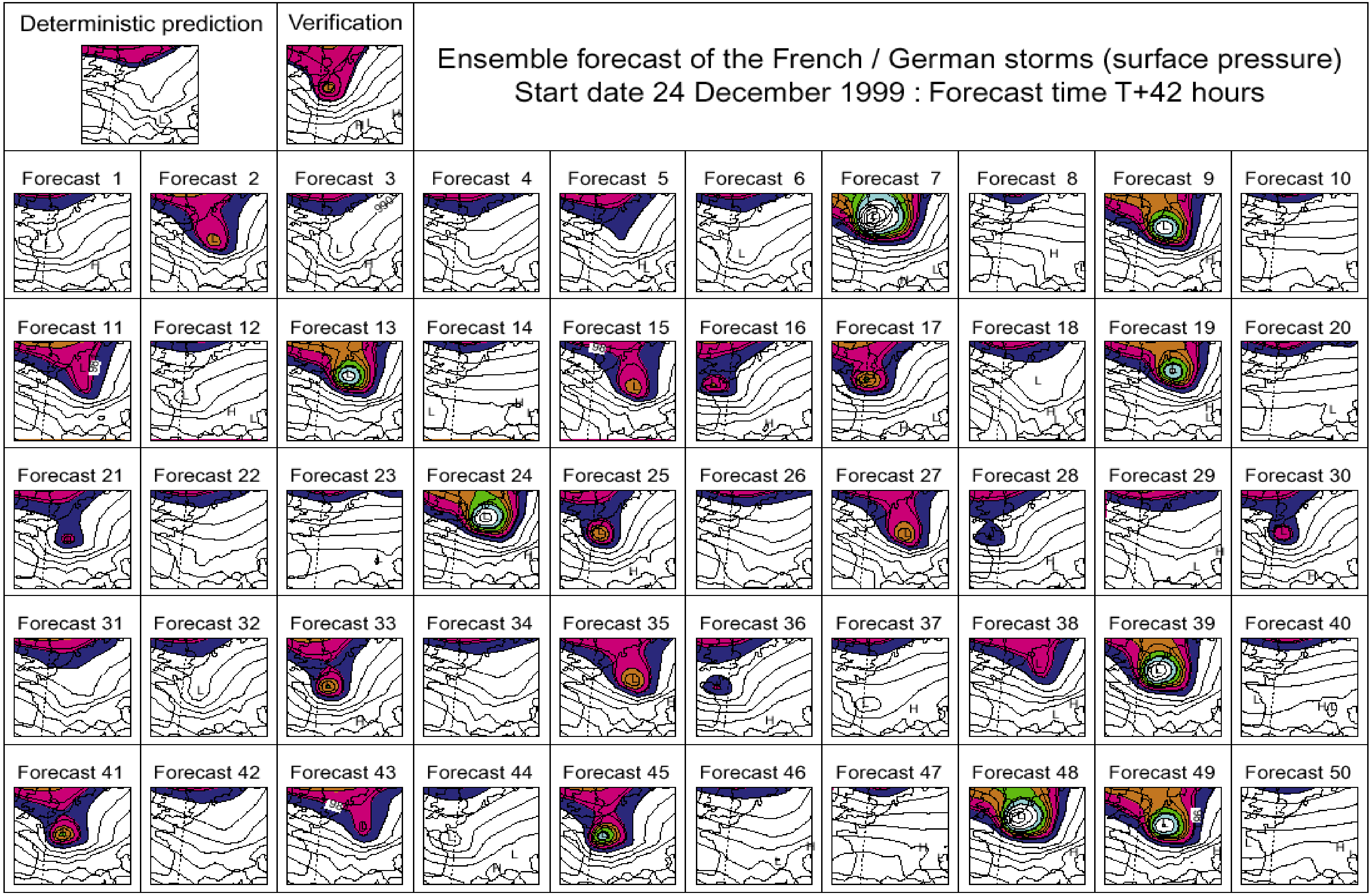


Ensemble Initial Conditions 24 December 1999

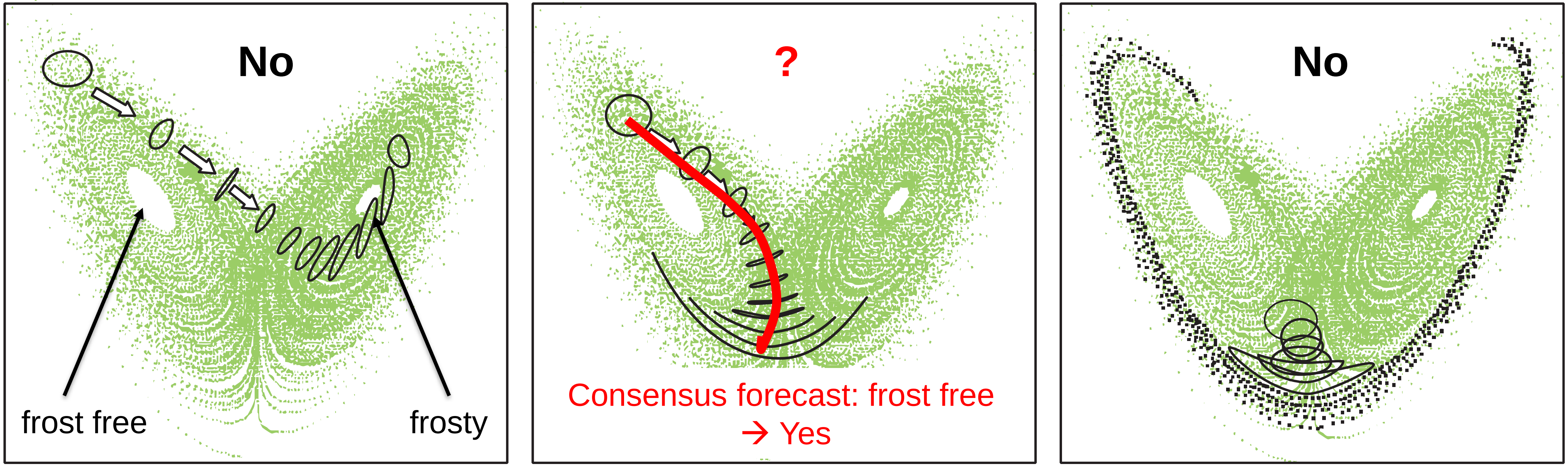




Lothar (T+42 hours)



Probabilistic forecasting and the cost-loss concept



Charlie is planning to lay concrete tomorrow. Should he?

Let ***p*** denote the probability of frost. Charlie loses **L** if concrete freezes. But Charlie also has fixed (e.g. staff) costs. There may be a penalty for late completion of this job - by delaying completion of this job, he will miss out on other jobs. These costs are **C**.

Is **L • *p* > C** ?

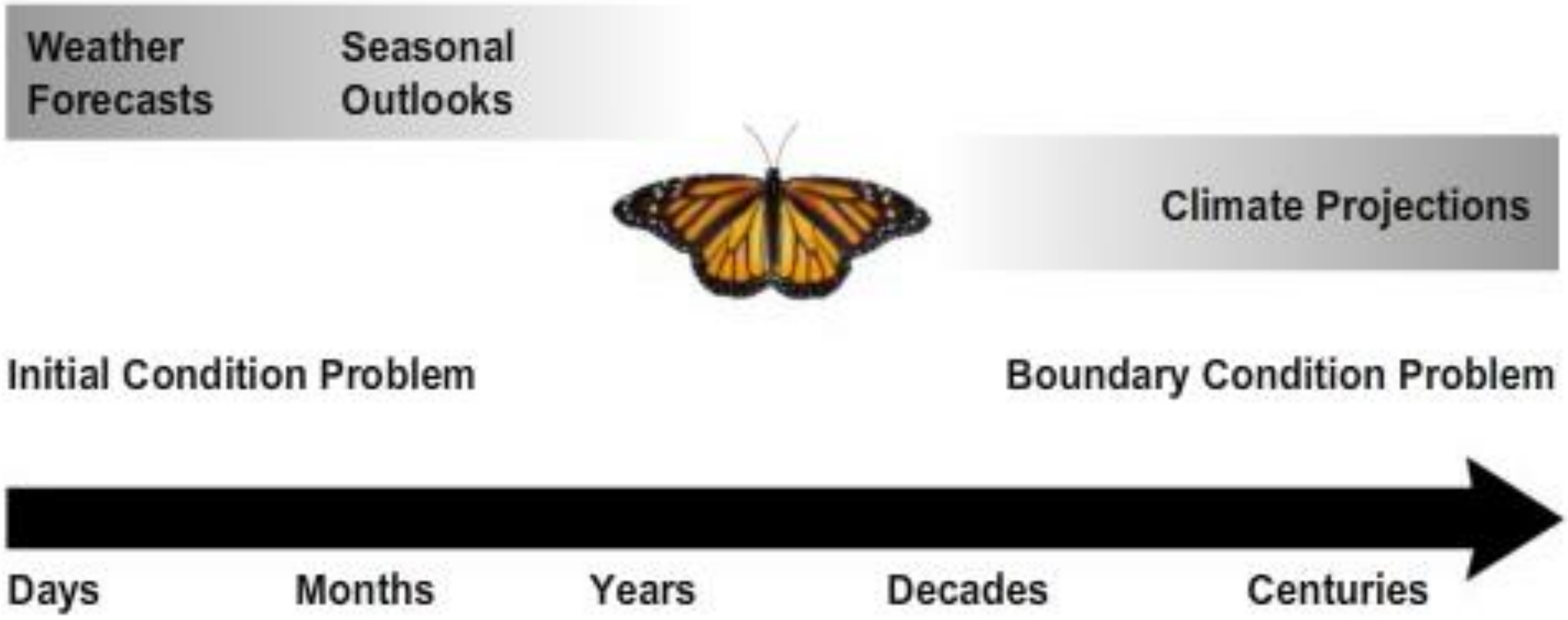
If *p* > C/L don't lay concrete!

Introduction to chaos for:

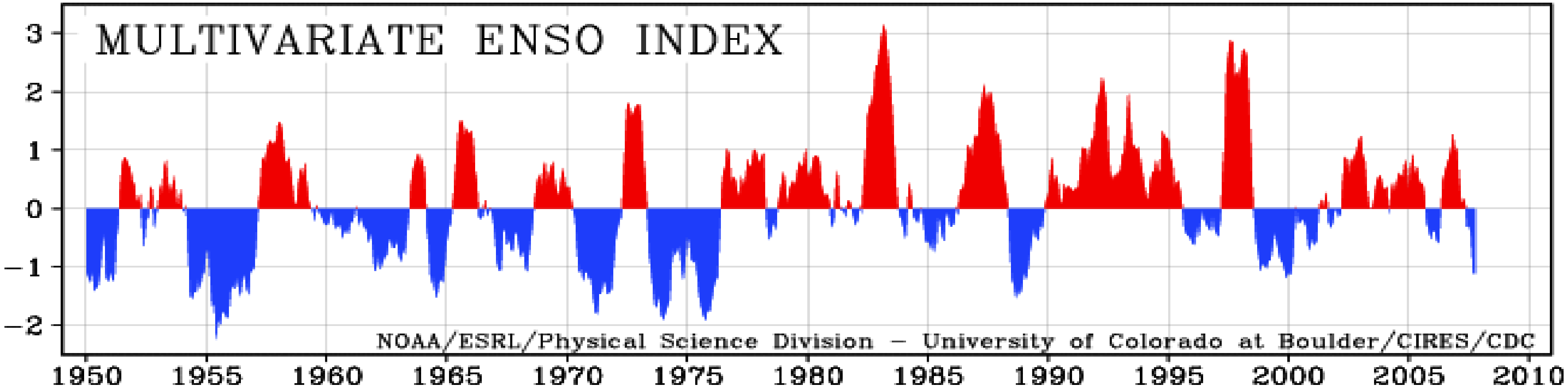
Seasonal climate prediction

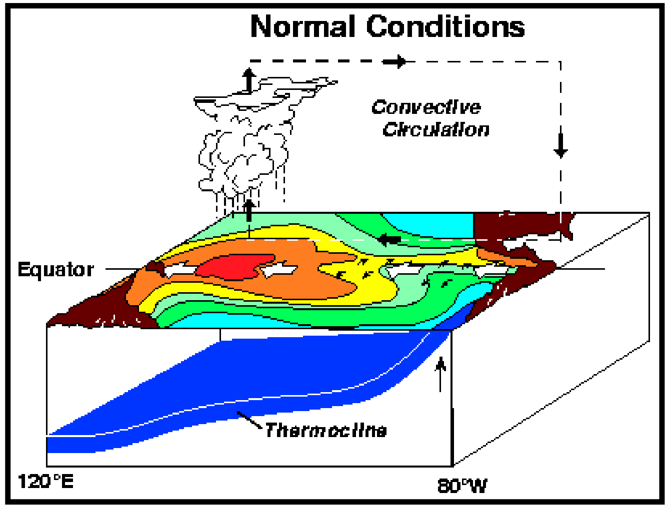
Atmospheric predictability
arises from slow variations
in lower-boundary forcing

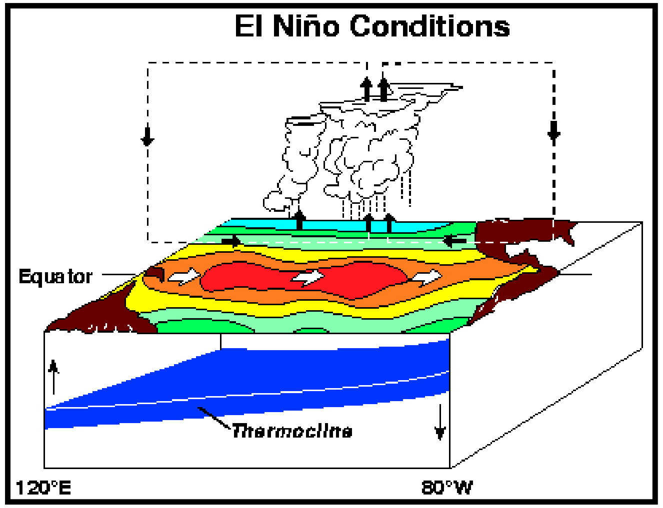
Climate forecasts are not crucially sensitive to the initial conditions. They are a mixed initial-boundary condition (forcing) problem in a chaotic system.

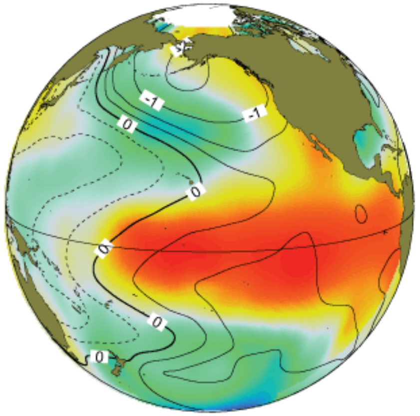


El Niño Southern Oscillation – a coupled atmosphere-ocean mode of variability

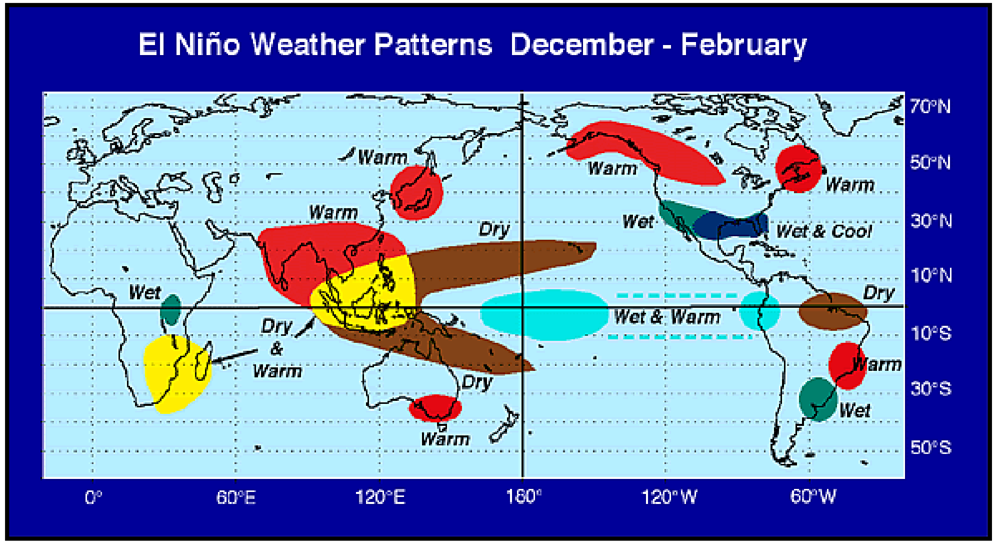


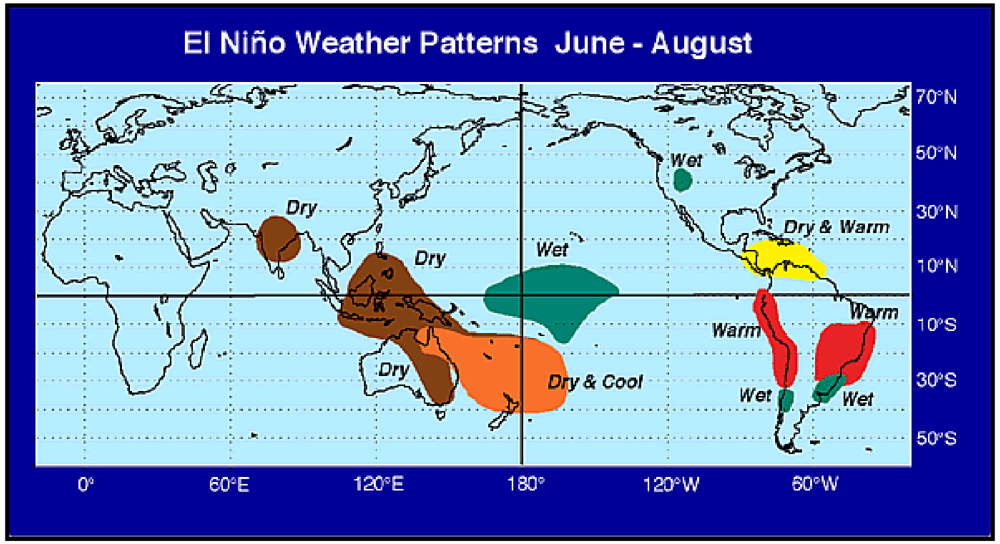




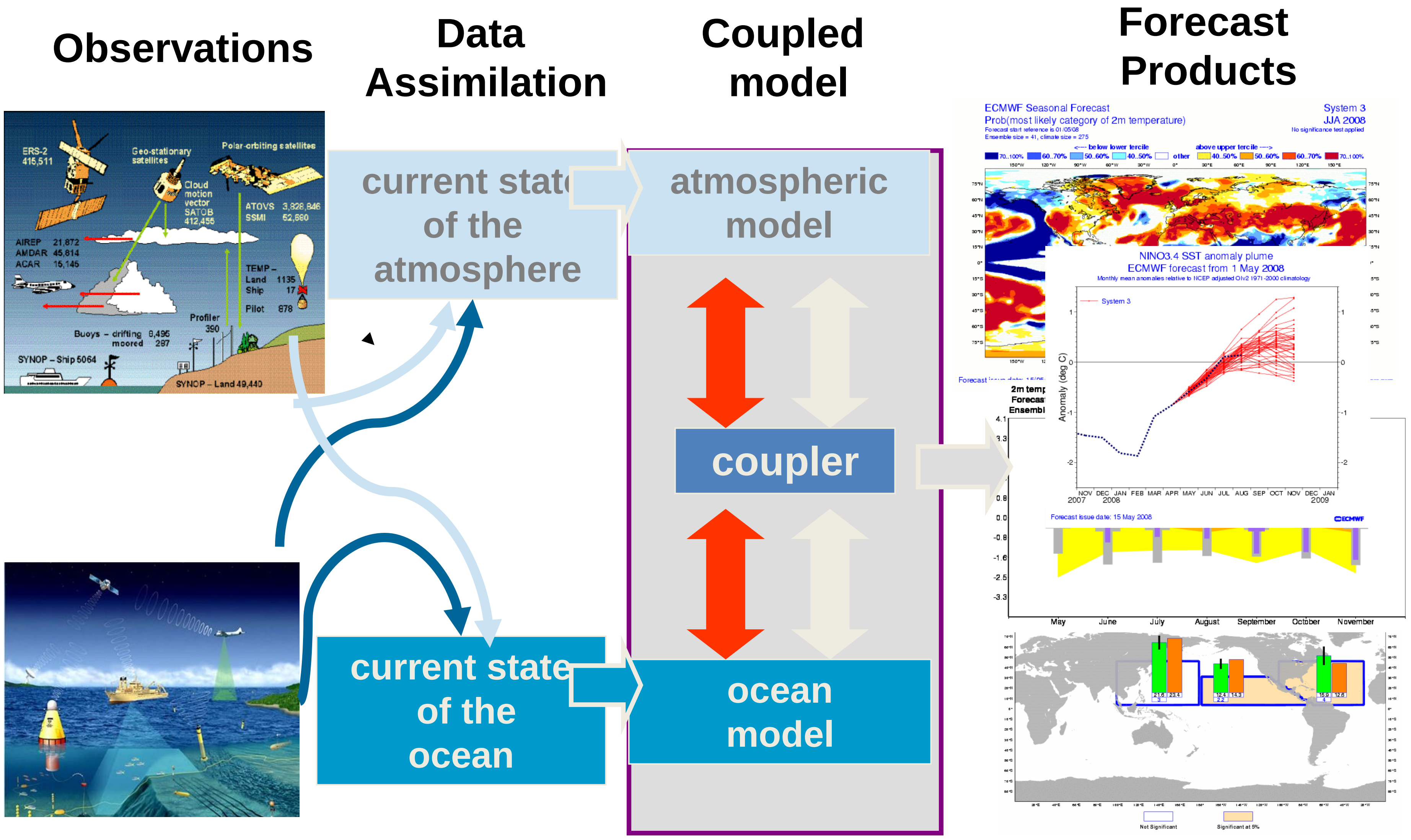


El Niño Southern Oscillation – a source of predictability on seasonal timescales

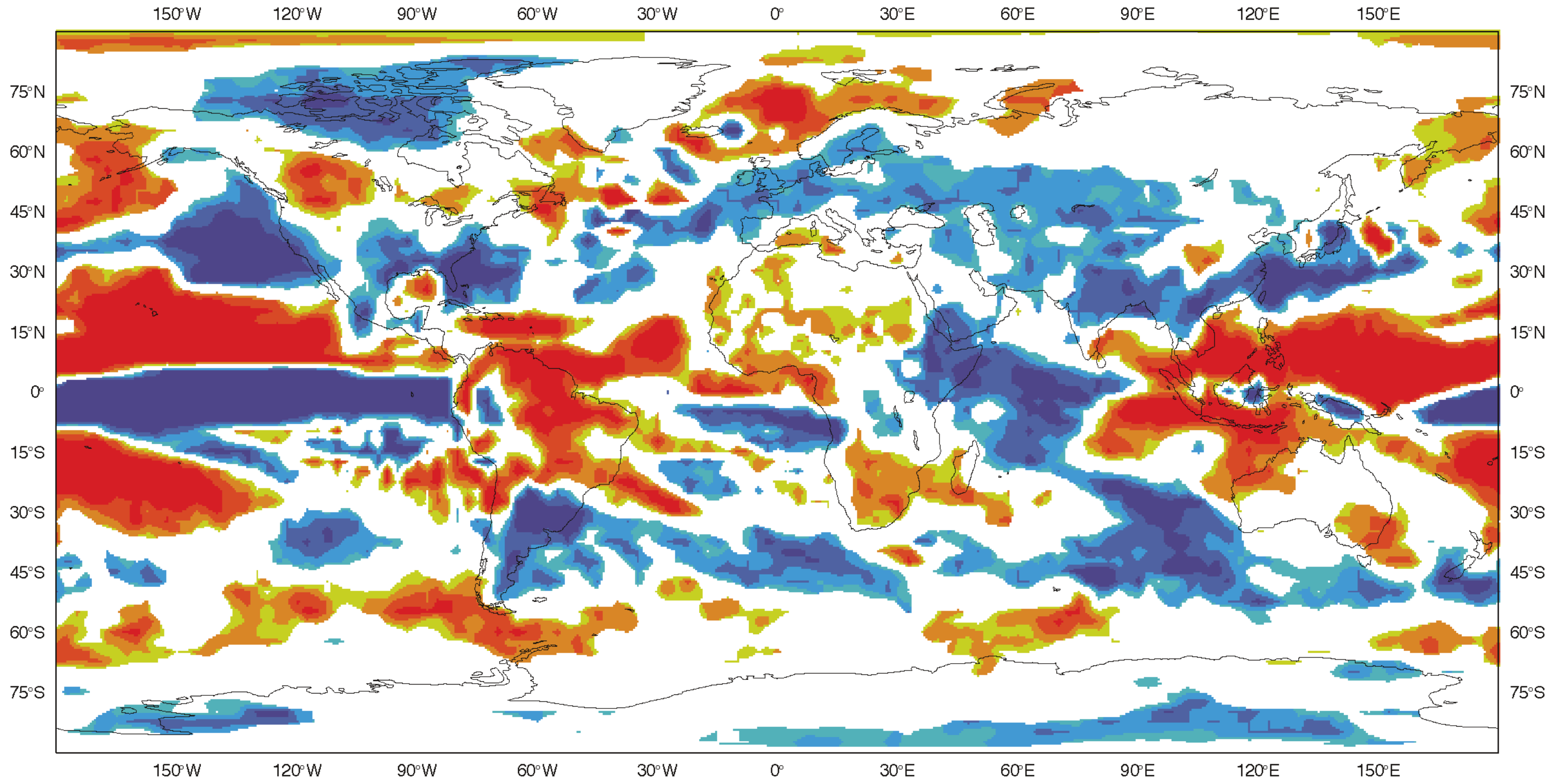




Forecast models for seasonal predictions



Seasonal Probability Forecasts (ECMWF / HOPE coupled model)
Winter 1997/98 from October 1997

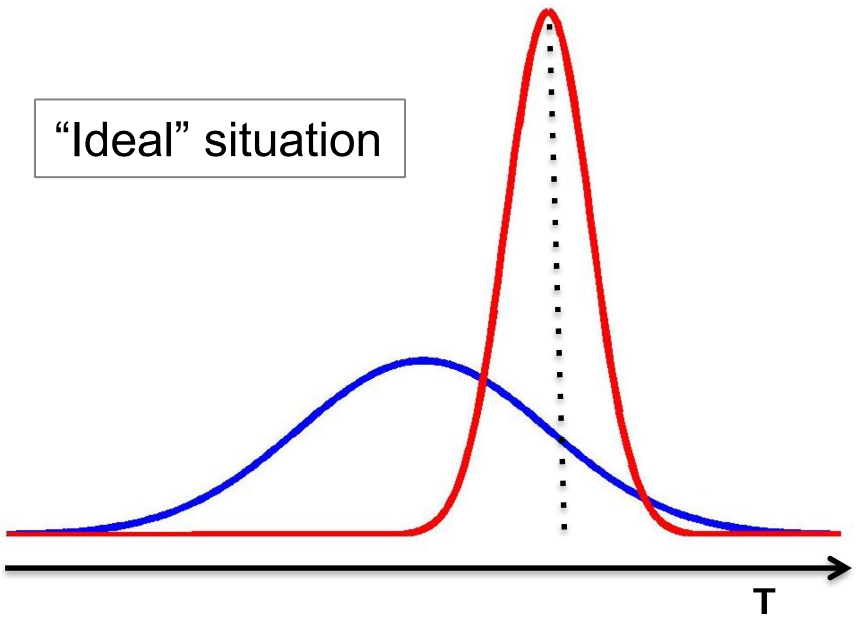


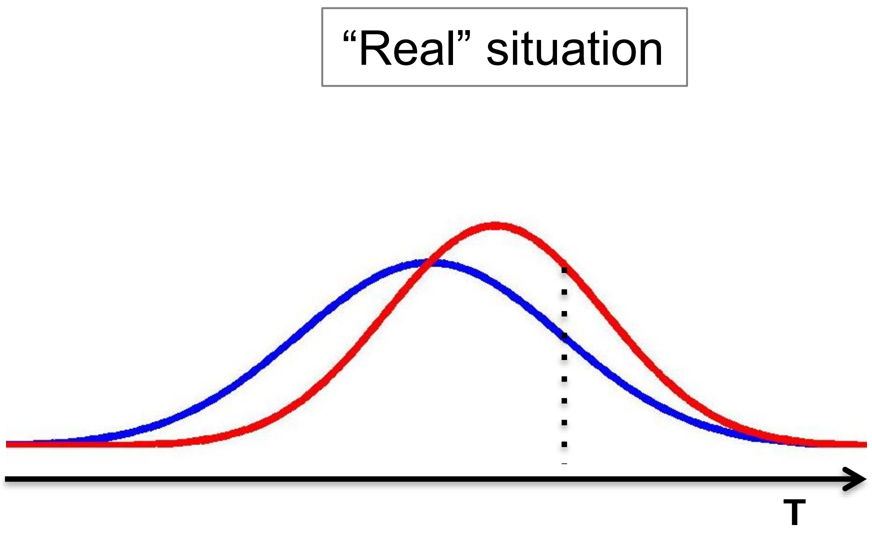
Blue: More likely to be wetter than normal, **Red:** More likely to be drier than normal

Forecasting probability distributions

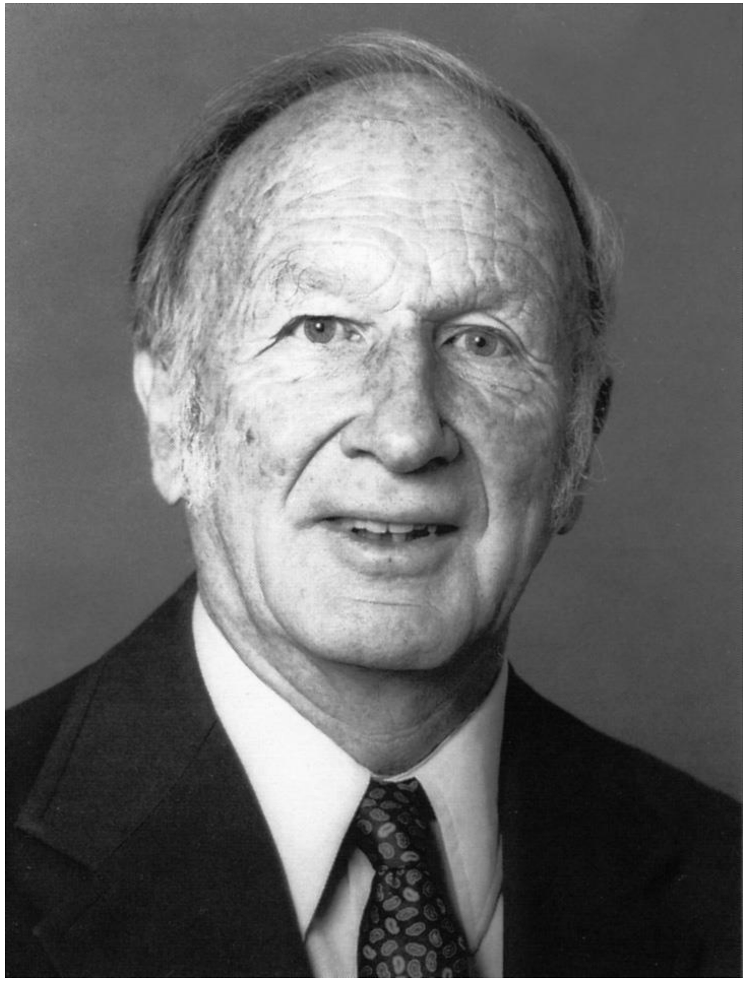
Seasonal forecasts aim to predict an anomaly from the default climatological probability.

Probability density distributions of a hypothetical climatology and forecast given an observation.





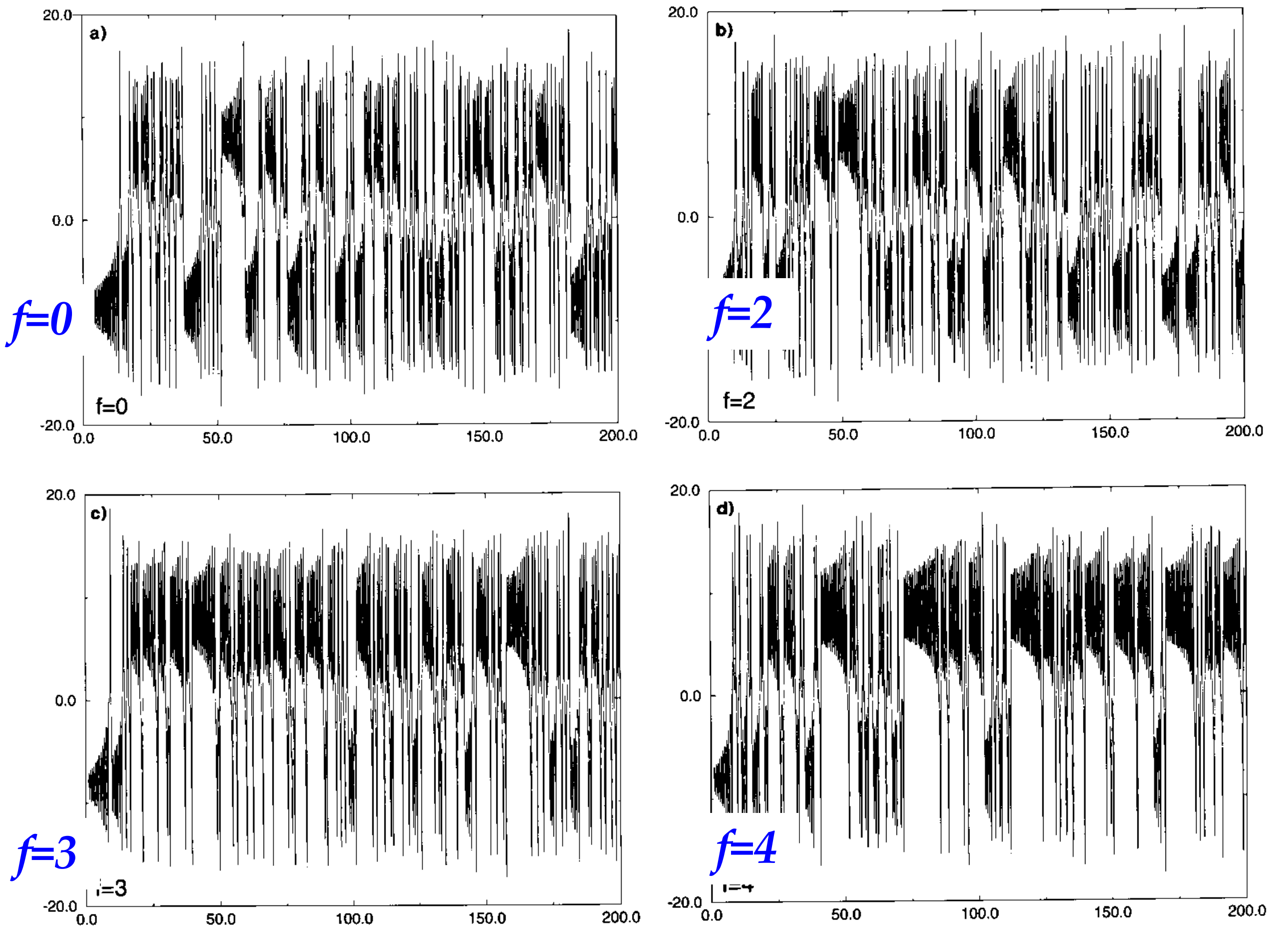
Edward Lorenz (1917 – 2008)



$$\dot{X} = -\sigma X + \sigma Y + f$$
$$\dot{Y} = -XZ + rX - Y + f$$
$$\dot{Z} = XY - bZ$$

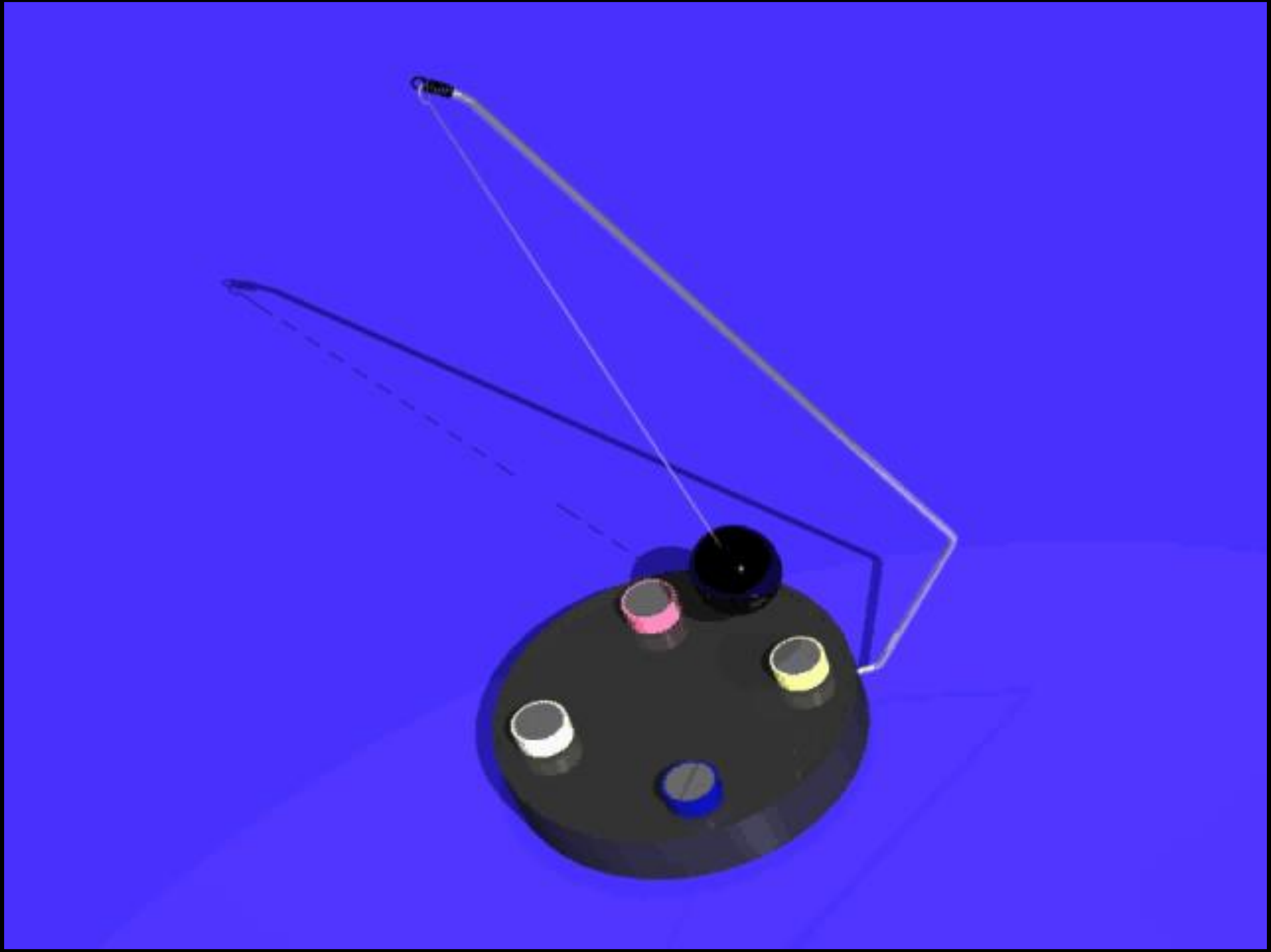
What is the impact of *f* on the attractor?

Add external steady forcing *f* to the Lorenz (1963) equations



The influence of *f* on the state vector probability function is itself predictable.

Mechanical analogue of preferred atmospheric circulation states



Preferred atmospheric circulation states: role of the forcing